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A CONSIDERATION OF THE NULL CLASS

Roger Osborn

It will be the purpose of this paper to pass over more or less briefly the following topics which are related to the null class: classes in general and their use and meaning, the history of the null class, definition of the null class and the universal class, arguments for and against the null class - its usefulness and its disadvantages, calculus of the null and universal classes, the indefinite class, and some other uses of the word null which are related to the null class. Arguments for and against the use of the null class in logic have been expressed by many competent men. It will not be the purpose of this paper to attempt to refute any of these arguments, but rather to present some of the more meritorious of them. The conclusion will be reached herein that the null class is a necessary component of modern logic.

A class has been described as a "totality of distinct objects which have some predicate in common. Every predicate determines such a totality." (VIII, p. 243) Rules of logical classification or division have been given as (XIII, p. 38) (*a*) every logical division must be based on the essential nature of the whole to be divided, (*b*) every logical division must have a single basis for division, (*c*) the species into which a class is divided must not overlap, and (*d*) the division must be exhaustive. *Nothing* and *Universe* are the two limits of class extension, for no possible interpretation of a class can relate to fewer individuals than none nor to more than are in the universe. The individuals of which a class is comprised are said to be members of the class, and they bear the relation of class membership. "Strictly speaking only a class can be *included* in a class, i.e., the relation of class inclusion is a relationship between classes, and not between an individual and a class." (VII, p. 264n.) It is important in class theory to recognize this distinction. Some of the difficulties arising from the use of the null class need not have been encountered if the authors had been more aware of this distinction. Russell says, "a class is said to exist if it has at least one member." (XVII, p. 21)

Before considering any of the technical aspects of the null class, it might be of interest to investigate its origins. The Greeks and Romans had no number zero in their mathematics, nor no null class in their logic. These classical logicians failed to provide for the contingency of empty classes. The importance of the null class was only recognized when the algebra of classes was developed. It has been argued that this departure from the traditional has devastated logic. There is some question as to the identity of the first person to introduce the null class into logic. Its introduction has been attributed variously to Leibnitz, DeMorgan, and Boole. Those who attribute its

introduction to Leibnitz include E. T. Bell (I, p. 517), mathematical historian, Louis Couturat (VI, p. vi), logician, and H. B. Smith (XVIII, p. 10), logician. Bell and Couturat claim that Leibnitz introduced what is now known as the null class; Smith claims only that Leibnitz envisioned the notion of a class with no member, hence necessitating a hypothesis of existence in order for Aristotelian logic to hold. Smith contends that DeMorgan introduced what is now known as the null class. Peano (XIV, p. 27) attributes the first use of the null class to Boole. Peano also has a comment to make on the use of the symbol Λ for the null class - the symbol used in many books. (XV, p. 7) He says that this is used because it is the inverted symbol V, used frequently for the universal class, which is the first letter of the French work "vrai", meaning "true", "real", or "genuine". For mathematicians, the second of these translations would probably be the meaning implied. It is known that Boole made extensive use of the null class in his *Laws of Thought.*

As an introduction to a consideration of the various definitions that have been given for the null class, it will be well to inquire into the necessity (questionable) of having a null class. Smith says (XVIII, p. 47) "We might want, for generality, for $a \times b$ to be a class if a is a class and if b is a class. But this requires the existence of a null class since some classes have no members in common. Also the denial of *everything* is *nothing*. Hence if there is a universal class, there must be a null class." Whitehead prefaced his consideration of the null class by writing (XX, p. 24), "On the assumption that to any question of the type of $a - b$ can be assigned an answer, some meaning must be assigned to the term $a - a$... Thus we put $a - a = 0$ ... "

Definitions of the null class have been given in many forms. In general they are alike, but some differ in enough respects to be given consideration. Following are some of the definitions given by various authors for the null class:

"a is said to be the null class if it satisfies the condition $a <$ not-a." (XVIII, p. 47)

"They (0 and 1) are defined by two axioms postulating their existence

Axiom: There is a term 0, such that whatever value may be given to the term x, we have $0 < x$.

Axiom: There is a term 1, such that whatever value may be given to the term x, we have $x < 1$." (VI, p. 17)

"The idea of a nonentity is indispensable to thought. Since the negatives of a concept are always significant, we can talk about nonentities if we can talk about entities. The null class is the class of nonentities." (VII, p. 421)

"A new feature of the class calculus is the null-class, or class having no terms. This may be defined as the class of terms that belong to every class, as the class which does not exist (in the sense defined above) [Russell says a class exists if it has at least one term], as the class which is contained in every class...

These can be shown to be equivalent." (XVII, p. 23)

"The symbol 0, as used in algebra, satisfies the following formal law, (1) $0 \times y = 0$ or $0y = 0$, whatever number y may represent. That this formal law may be obeyed in the system of logic, we must assign to the symbol 0 such an interpretation that the class represented by 0y may be identical with the class 0, whatever the class y may be. A little consideration will show that this condition is satisfied if the symbol 0 represent *nothing*. In accordance with a previous definition [of class], we may term Nothing a class. In fact, Nothing and Universe are the two limits of class extension, for they are the limits of the possible interpretations of general names, none of which can relate to fewer individuals than are comprised in Nothing, or to more than are comprised in the Universe. Now whatever class y may be, the individuals which are common to it and to the class *Nothing* are identical with those comprised in the class *Nothing*, for they are none. And thus by assigning to 0 the interpretation Nothing, the law (1) is satisfied; and it is not otherwise satisfied consistently with the perfectly general character of the class y." (IV, p. 52)

"There is also a null class, one that has no members, the class of *nothing*; and this class is determined by any predicate of which there are no instances." (VIII, p. 243)

"Classes which have no members are called *empty classes*. If they are determined by a self-inconsistent concept, so that they could have no members, they are said to be not merely empty but also *null*. A null class is sometimes called a *vacuous* class. Every null class is empty, but not all empty classes are null." (II, p. 102)

"An *empty* class is one containing no members, e.g. the class of round squares... An empty class is sometimes called a 'null-class', but as the adjective 'null' has been appropriated by writers on Analysis, another word is necessary." (IX, p. 2)

"According to Jevons, 0 indicates that which is contradictory or 'excluded from thought'. This is its intensional meaning. In extension it is the null class." "The null class may be defined as the negative of the universal class." (X, p. 73, p. 263)

"The null-set, denoted by zero, has no members and is contained in every set... It follows from the definition of the null set that a necessary and sufficient condition that two sets A and B meet is $AB \neq 0$." (XII, p. 5)

"Let the elements of the algebraic manifold be regions in space... The null element must be interpreted as denoting the non-existence of a region. Thus if a term represent the null-element, it symbolizes that the mind after apprehending the component regions (if there be such) symbolized by the term, further apprehends that the region placed by the term before the mind for apprehension does not exist. It may be noted that the addition of terms which are not null cannot result in a null term. A null term can however arise in the multiplication of terms which are not null." (XX, p. 38)

It has been stressed by many writers that the null class is unique. These expressions have taken many forms, and the arguments are varied. Eaton (VII, p. 213n.) argued that, "There is only one null class – a class with no members could not be distinct from another class with no members since where there is no distinction of members there is no distinction of classes." It has also been argued that class properties depend upon the extension of the members, and hence there is only one class *nothing*, and all terms which name no existent thing denote the same class – the null class. (XI, p. 28) Whitehead, however, does not advance this argument. In fact he has said (XX, p. 24), "It would be wrong to think of 0 as necessarily symbolizing mere nonentity. For in that case, since there can be no differences in nonentities, its equivalent forms $a - a$ and $b - b$ must be not only equivalent, but absolutely identical; whereas they are palpably different." Whitehead could have been led to no other conclusion from his definition of the null class (given above), though it is questionable whether $a - a$ and $b - b$ are in fact different.

From these definitions and properties of the null class of logic, a logical concept of the number zero has arisen in mathematics. This is not meant to imply that zero in mathematics is defined in the same way as the null class nor that it is defined at all. Peano showed that the entire theory of natural numbers could be derived from three primitive ideas and five primitive propositions in addition to those of pure logic. The three primitive ideas were 0, member, successor. One of the five primitive propositions was that 0 is a number. (XVI, p. 5) Frege defined the number zero as the number of terms in a class which has no members. Russell says, "zero is the class whose only member is the null class." (XVI, p. 23) Zero is not identical with the null class since it (zero) has one member, namely, the null class, whereas the null class itself has no members.

Since so many references have been made to what Bertrand Russell has written, it should be mentioned in passing that he also said, "There is no such thing as the null class, though there are null class-concepts." (XVI, p. 68) He relegates these concepts to a realm of subsistence rather than existence.

Another interesting relation between propositions and the null class has been worked out. Among other expressions of this relation is that of Whitehead (XX, p. 109): "The null element of the manifold of algebra corresponds to the absolute rejection of all motives for assent to a proposition, and further to the consequent rejection of the validity of the proposition. Hence, $x = 0$ comes to mean the rejection of x from any process of reason, or from any act of assertion. In so far as they are thus rejected all such propositions are equivalent. Thus if $x = 0$, $y = 0$, then $x + y = 0$. Furthermore if $b = 0$, the proposition $a + b$ is equivalent to the proposition a alone; for the motives of validity of b being absolutely rejected, those for the validity of a alone remain. Again if $b = 0$, then $ab = 0$, for ab means that a and

b are asserted conjointly, and if the motives for *b* be rejected, the motives for the complex proposition are rejected. It is now possible to define the class, necessarily of indefinite number, of propositions which are to be equated to the null element. This equation must not rest merely on the empirical negative fact of the apparent absence of motives for assent; but on the positive fact of inconsistency with the propositions which are equated to the Universe. If the Universe be reduced to the Laws of Thought, then all propositions equated to null are self-contradictory. With a more extended Universe, all propositions equated to null are those which contradict the fundamental assumptions of our reasoning."

There have been arguments presented for and against the use of the null class in logic. One argument used frequently is that it corresponds to the zero in mathematics, and that the use of zero in mathematics has proved to be of value. A mere analogy, though, is insufficient evidence that it should be used. Its usefulness in symbolic logic can hardly be denied, but logicians who yet cling to the belief that traditional logic is the ultimate in perfection, and that modern innovations have adulterated that which was perfect, still claim that there is no place for the null class in logic. Even many of the symbolic logicians can see difficulties arising from its use. One of the better arguments given for its inclusion in logic is given by Bennett and Baylis (II, pp. 103-4): "In discussing operations upon classes we find that, if no null class is conceded to exist, then the phrasing of the laws of class relations becomes complicated, owing to the special cases and exceptions, which cause difficulties that disappear under a meaning of the term *class* broad enough to permit empty classes." "In every argument by *reductio ad absurdum*, one draws conclusions from hypotheses entertained but shown eventually to lead to inconsistency. One argues, in short, from self-inconsistent properties, and therefore, one may say, from classes which are null. Only those who never use indirect proofs are privileged to reject the validity of arguments based on inconsistency and the null class." The substance of most arguments against using the null class is that its use devastates the traditional relations between propositions on the square of opposition. In addition to this type of argument, it has also been argued that if the null class is the lower limit of the instances of a class, then it probably should not be called a class, but a nonentity. Most modern logicians have been able to overcome all these difficulties by suitable limitations and suitably chosen procedures. Notable among these has been Bertrand Russell. He recognized many difficulties, but finally came to the conclusion that, "The null class...cannot be interpreted on the same principles as other classes..." but that the null class "may be admitted." (XVII, p. 75, p. 106)

More or less as an added commentary on the null class, it should be noted that Boole considered, in addition to the null class, an

indefinite class. He wrote, "In arithmetic, the symbol $\frac{0}{0}$ represents an indefinite number, except when otherwise determined by some special circumstance. Analogy would suggest that in the system of logic, the same symbol would represent the indefinite class." (IV, p. 96)

Before leaving the subject of the null class, some mention should be made of the calculus of the null class (and the universal class, too, since their algebraic properties are very much alike). Axioms postulating their existence have already been given. The following formulas comprise the rules of the calculus for 0 and 1 (the universal class):

$$a \times 0 = 0 \qquad a + 1 = 1$$

$$a + 0 = a \qquad a \times 1 = a.$$

From these rules and from the definitions of 0 and 1, the following assertions may be made:

(1) It does not change a term to add 0 or multiply by 1. Hence 0 is called the modulus of addition and 1 of multiplication. (VI, p. 17)

(2) To say that a sum is null is to say that each summand is null.

(3) A class contained in the null class is the null class itself.

(4) The addition of two terms which are not null cannot result in a null term.

(5) The multiplication of two terms which are not null can result in a null term.

(6) If 0 and 1 exist, then for any class a there is a class $-a$ such that $a + -a = 1$ and $a \times -a = 0$.

(7) By definition 0 is included in 0, 0 is included in 1, and 1 is included in 1. The first and last of these result from the principle of identity.

These seven assertions by no means exhaust the possible conclusions which may be drawn from the definitions and the principles of operation on these symbols.

Finally, some of the uses of the word *null* will be given. These uses arise from its meaning as used in *null class*, but this meaning has been carried over into many fields other than formal symbolic logic. The following uses come from both logic and mathematics.

(1) Any term such as $a - a$ is called a null term. (XX, p. 24)

(2) A propositional function is said to be null when it is false for all values of x. (XVII, p. 22)

(3) A proposition which is the contrary of itself is called a null-proposition. (XVIII, p. 83)

(4) A null-property is a property not possessed by any object. (XII, p. 5)

(5) If the magnitude $|A|$ of the vector A is zero, then the vector is called the null vector, and in this case the notion of direction is meaningless.

(6) The matrix algebra consisting solely of the one-rowed matrix 0 or the abstract algebra consisting of the single element 0 is called the null-algebra. (XIX, p. 85)

(7) A circle with zero radius is a null-circle.

(8) A null sequence is a sequence which converges to the limit zero. (V, p. 446)

(9) The null-space of a linear transformation T is the set of all vectors ξ such that $\xi T = 0$. (III, p. 268)

(10) If the product ab of two elements of a ring never vanishes unless at least one of the factors vanishes, the ring is without *null-divisors*. (XIX, p. 2)

In conclusion, these observations should be made. The null class, though possibly being the class of nonentities, is not itself a nonentity. It does not exist – in the sense of material existence – but it does exist in the realm of concepts. It is not only convenient in logic, but it is a necessary part of modern logic, for without it a logical algebra – or an algebra of logic – could not exist in a large enough sense to be useful.

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ON METRIC REPRESENTATIONS OF GROUPS*

David Ellis

1. *Introduction.* Garrett Birkhoff and Robert Frucht have shown how to represent abstract groups as groups of automorphisms on partially ordered sets and lattices[1]. Also, A. Fröhlich[2] has considered the representation of finite groups as groups of automorphisms on finite Abelian groups. These studies suggest the question: *What type of group may be represented as a group of motions on some metric space*[3]? By a very elementary construction, this note provides the answer to the above question for finite and countable groups in the rather fascinating result: *Any finite or countable group is a group of motions on a metric Baire space.*

2. *Metric Baire space.* This space is well-known in the literature but we give its construction here for convenience. Let p_1, p_2, ... be any countable set C of elements (points). Let $\mathfrak{B}(C)$ be the set of all (countable) sequences of elements formed from the set C. For $\mathfrak{a} = (a_1, a_2, \ldots)$ and $\mathfrak{b} = (b_1, b_2, \ldots)$ in $\mathfrak{B}(C)$ define $\delta(\mathfrak{a}, \mathfrak{b}) = 1/k$ where k is the first integer for which $a_k \neq b_k$. Define $\delta(\mathfrak{a}, \mathfrak{a}) = 0$. It is easily verified that $\mathfrak{B}(C)$ forms a metric space under $\delta(\mathfrak{a}, \mathfrak{b})$. It is called the *metric Baire space* formed on C. One applies precisely the same procedure to obtain a space $\mathfrak{B}$ on a finite set p_1, ..., p_n. Here, of course, it suffices to consider only sequences of n elements. In this case, non-zero distances are bounded below by $1/n$.

3. *The theorem.*

Theorem. *Let G be a finite or countable group. G is (isomorphic to) a subgroup of the group of motions of the metric Baire space $\mathfrak{B}(G)$.*

Proof. We consider G countable. The finite case is similar (we observe also that one might first imbed a finite group in a countable group and obtain the finite group as a group of motions on an infinite Baire space). Form $\mathfrak{B}(G)$. Let $a \in G$ and $\mathfrak{x} = (x_1, x_2, \ldots) \in \mathfrak{B}(G)$. Define $a(\mathfrak{x}) = (ax_1, ax_2, \ldots)$. Then $a(\mathfrak{x})$ is a mapping of $\mathfrak{B}(G)$ into itself. It is biuniform due to the non-singularity of group multiplication.

*Presented to the American Mathematical Society; Christmas, 1951.

[1] See Garrett Birkhoff, *Sobra los grupos de automorfismos*, Revista Union Mat. Argentina, vol. XI, pp. 155-157; Robert Frucht, *Sobre la construccion de sistemas parcialmente ordenados con grupo de automorfismos dado*, Revista Union Mat. Argentina, vol. XIII, pp. 12-18; Robert Frucht, *On the construction of partially ordered systems with a given group of automorphisms*, Am. Jour. of Math., vol. LXXII, pp. 195-199.

[2] A. Fröhlich, *The representation of a finite group as a group of automorphisms on a finite Abelian group*, Quart. Jour. of Math. (Oxford (2)), vol. 1, pp. 270-283.

[3] A *motion* of a metric space is an isometric mapping of the space onto itself.

It is also *onto* $\mathfrak{B}(G)$ since $\mathfrak{x} = (x_1, x_2, \ldots)$ is the image of $(a^{-1}x_1, a^{-1}x_2, \ldots)$. Since the multiplication of G is associative, $a(b(\mathfrak{x})) = ab(\mathfrak{x})$ and the group product of a and b agrees with their mapping product. Thus, G is a group of biuniform mappings of $\mathfrak{B}(G)$ onto itself. Let $\delta(\mathfrak{a},\mathfrak{b}) = 1/k$. Then $a_i = b_i$; $i < k$ and $a_k \neq b_k$. We have $aa_i = ab_i$; $i < k$ and $aa_k \neq ab_k$ so that $\delta(\mathfrak{a},\mathfrak{b}) = \delta(a(\mathfrak{a}), a(\mathfrak{b}))$ and a is a motion of $\mathfrak{B}(G)$.

Remark. The major inference to be drawn from the above theorem is that the algebraic structure of groups of motions on a metric space apparently has little to do with determining the geometric structure of the space. Thus, a countable subgroup of the rotation group on Euclidean three-space is also a group of motions on a Baire space which is extremely different not only metrically but topologically from Euclidean space.

Remark. Of course, any two metric Baire spaces formed on countable sets are isometric so that we have also from the theorem that the group of motions of the metric Baire space formed on a countable set contains all countable and finite groups as subgroups.

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A NOTE ON THE POISSON LAW

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Introduction – The Poisson law states that the probability $p_n(x)$ of n points in an interval of length x is

$$(1) \qquad p_n(x) = (kx)^n e^{-kx}/n!$$

when the points are distributed "individually and collectively at random". Since $\Sigma np_n(x) = kx$, the constant k is the expected number of points in a unit interval. A good working definition of individual and collective randomness is given in [4], which we quote:

'A set of points is said to be distributed "individually at random" along a line segment provided each point of the set is placed at random, independently of all the rest.

'A set of points is said to be distributed "collectively at random" along a line segment provided the probability of any interval dx containing n points is independent of the number of points in any interval not wholly or partly included in dx.'

For collective randomness various other formulations can be given. Essentially, it means that there is no bunching, [4], and is sometimes taken as equivalent [1], [2], [4] to $(1/x)\sum_{2}^{\infty} p_n(x) \to 0$ as $x \to 0$.

Now, a set of points can be collectively random and yet not individually random. An approximation to this situation is given by the following example [4]: Imagine that the "points" are people on a highway. For isolated pedestrians one can fairly suppose both individual and collective randomness. But from time to time there is a bus! Surely the passengers are not placed "independently of all the rest"; on the contrary, each goes where the others do. Hence they are not distributed individually at random. Still, they are distributed collectively at random, if the busses are.

Does the Poisson law hold in such a situation? To answer this question the counterexample $p_{2n+1}(x) = 0$, $p_{2n}(x) = (kx)^n e^{-kx}/n!$ is well known. Points so distributed are collectively but not individually at random; and this distribution is essentially different from (1).

What we really have here is a Poisson distribution of point pairs. More generally one can construct examples with point triplets, quadruplets, $\cdots$, or any combinations of these. Thus we are led back to Fry's example of the highway. The "pairs" are cars with two passengers, the "triplets" are cars with three, and so on. It is natural to ask whether every distribution with collective randomness can be attained by combining Poisson distributions in this way. The answer is yes; the demonstration is the chief purpose of the present note.

Objectives and results – First we characterize the distributions that have collective randomness:

Theorem I. Suppose points are distributed collectively at random and that the probability of having infinitely many points in any finite interval is zero. If $p_n(x)$ is the probability of n points in an interval of length x, then $\sum p_n(x)z^n = e^{xa(z)}$, where the function $a(z)$ has a convergent expansion $a(z) = -s + az + bz^2 + cz^3 + \cdots$ near $z = 0$, with $a, b, c, \cdots$ all ≥ 0, and with $s = a + b + c + \cdots$. Also every function $a(z)$ of this type leads to a sequence of probabilities $p_n(x)$ satisfying the conditions of the hypothesis.

The function $e^{xa(z)}$ in the theorem is the so-called generating function of the sequence $p_n(x)$; it is the discrete analogue of the characteristic function. Once $a(z)$ is known, $p_n(x)$ can be found (at any rate in principle) by expanding and comparing coefficients. For example when $a = k$, $b = c = \cdots = 0$, we have $a(z) = -k + kz$, $e^{xa(z)} = e^{-kx}e^{kz} = \sum e^{-kx}(kz)^n/n!$, so that (1) follows. As is often the case, the generating function is far simpler than $p_n(x)$ itself.

Theorem I, then, characterizes all collectively-random distributions. It remains to show that they can be realized by a suitable combination of distributions each of which has both individual and collective randomness:

Theorem II. Suppose that the function $a(z)$ in I has expansion $a(z) = -s + az + bz^2 + cz^3 + \cdots$. If we distribute single points by a Poisson distribution (1) with parameter $k = a$, point pairs by (1) with parameter b, and so on, then the probability of n points in an interval of length x is the $p_n(x)$ of Theorem I.

From this one can see the true meaning of collective randomness: If the expected numbers of point pairs, triplets, $\cdots$, are zero then the Poisson distribution holds, and otherwise it does not hold. Also every collectively-random distribution can be obtained in the manner suggested by Fry's traffic example quoted above.

These results form our major objectives, but in the course of the proof we obtain certain minor objectives. First, the proof in [4] has several points at which the argument is not rigorous, and at which it cannot easily be made so. A rigorous proof is given in [1], [2]; but there the definitions are formulated in such a way that one of the essential desiderata seems to be included in the hypothesis. Here we define individual and collective randomness by the verbal statements quoted from [4], with no additional assumption whatsoever. From this the result is to be deduced in all rigor.

A second minor objective is to illustrate the use of generating functions in this connection. The proof in [4] is much simplified by such an approach, but here we exploit the method more fully. So far as we know the results and proof are both new, at least for this level of generality; we have not seen the method of generating functions applied before to the derivation of (1), nor have we come across the particular

representation theorems I, II. Any readers familiar with the infinite divisibility laws and the compound Poisson process will recognize certain of the present results as special cases of very general theorems. But even for such readers it may be interesting to see how far one can go without the heavy use of characteristic functions, continuous distributions and advanced analysis that usually accompanies their discussion.

Proof of Theorem I – As in [4] we write

$$p_n(x + y) = \Sigma p_{n-k}(x)p_k(y) \tag{2}$$

which follows from the hypothesis and from the fact that we can have n points in $x + y$ only by having $n - k$ in x and k in y, for some k. If $f(z,x)$ is the generating function

$$f(z,x) = \Sigma p_n(x)z^n, \tag{3}$$

then (2) gives us

$$f(z,x + y) = f(z,x)f(z,y). \tag{4}$$

We want to show that $f(z,x)$ is a continuous function[1] of x for $|z| < 1$. To this end we show first that $p_0(x) \to 1$ as $x \to 0$. Suppose on the contrary that $p_0(\delta) \leq \theta < 1$ for infinitely many arbitrarily small δ. Then divide the interval x into m small intervals of length δ, use the fact that $p_0 \leq 1$ to account for the overlapping at the ends, and note that $p_0(x) \leq [p_0(\delta)]^m \leq \theta^m$, which can be made arbitrarily small by picking δ small enough. Since $p_0(x)$ is independent of δ, we must have $p(x) = 0$. Thus every interval has infinitely many points, with probability 1; and this contradicts the hypothesis.

We may suppose then that $p_0(x) \to 1$ as $x \to 0$. Since $\Sigma p_n(x) = 1$, and $p_n(x) \geq 0$, it follows that $p_n(x) \to 0$ as $x \to 0$ for $n \geq 1$. In view of these results, Eq. (2) gives $\lim_{y \to 0+} p_n(x + y) = p_n(x)$, and with $u = x + y$ Eq. (2) also gives $p_n(u) = \lim_{y \to 0+} p_n(u - y)$. Hence $p_n(x)$ is continuous, $n = 0, 1, 2, \cdots$. Continuity of $f(z,x)$ in (3) follows by uniform convergence of the series.

The only continuous solution of (4) is the exponential, so that

$$f(z,x) = e^{xa(z)}. \tag{5}$$

For small x we know that $p_0(x)$ is near 1, therefore not zero; and thus $f(0,x) \neq 0$. Hence $a(z)$ in (5) is regular about $z = 0$.

[1]The reader may want to skip over the proof of continuity in this and the following paragraph. It is rather tedious, and adds nothing to the leading ideas. Actually one can sidestep the whole discussion if one is willing to assume the deeper result that the exponential is the only bounded solution of (4) (as well as the only continuous solution).

By induction it is found that the n^{th} derivative of $e^{xa(z)}$ is of the form $[(a^{(1)})^n x^n + \cdots + a^{(n)}x]e^{xa(z)}$, which has the sign of $a^{(n)}$ for small x. Hence $p_n(x) \geq 0$ makes $a^{(n)}(0) \geq 0$ for $n \geq 1$. That $a(z) \to 1$ as $z \to 1-$ follows from $\Sigma p_n(x) = 1$ and uniform convergence; and $s = a + b + c + \cdots$ follows from this, since the coefficients are positive.

For the converse, suppose given $a(z)$ regular at $z = 0$. Then $e^{xa(z)}$ may be expanded, and the above shows (2) will hold. From this it is easy to deduce individual randomness, provided $p_n(x)$ are probabilities. But the other conditions on $a(z)$ ensure $p_n(x) \geq 0$, $\Sigma p_n(x) = 1$; hence $0 \leq p_n(x) \leq 1$.

We have incidentally shown that $p_n(x) = A_n(x)e^{-kx}$ where A_n is a polynomial of degree n with non-negative coefficients, satisfying $\Sigma A_n(x) = e^{kx}$ and an equation of form (2). Also any such set of A_n's is permissible. The case $a(z) = -s + az$ or $A_n(x) = a_n x^n$ reduces immediately to (1).

Proof of Theorem II – To prove II suppose there are α single points, β pairs, γ triplets, and so on, in an interval of length x. The probability of this contingency is

$$(a^\alpha/\alpha!)(b^\beta/\beta!)(c^\gamma/\gamma!)\cdots x^{\alpha+\beta+\gamma+\cdots}e^{-(a+b+c+\cdots)x}$$

by independence and (1); hence the probability of n points is the sum subject to $\alpha + 2\beta + 3\gamma + \cdots = n$. On the other hand this same expression is the coefficient of z^n in $e^{-sx}\Sigma(x^k/k!)(az + bz^2 + cz^3 + \cdots)^k$ as we see by the multinomial theorem, and by recalling that $s = a + b + c + \cdots$.

The result can also be found from the fact that the characteristic function of a sum is the product of the characteristic functions, when we replace "characteristic function" by "generating function" and write

$$e^{xa(z)} = e^{-sx}e^{xaz}e^{xbz^2}e^{xcz^3}\cdots$$

It is interesting that the requirement concerning infinitely many points in Theorem I could be dispensed with; one would then obtain a theorem of the same type with $s \geq a + b + c + \cdots$, instead of $s = a + b + c + \cdots$. The probability of an interval's containing 0 or 1 or 2 or 3 $\cdots$ points would now be less than 1, the remaining case being that it has infinitely many. The representation II remains valid, with minor modification.

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COLLEGIATE ARTICLES

Graduate training not required for reading

SOME APPLICATIONS OF FINITE DIFFERENCES

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Foreword

Cases sometimes arise in which the equations of curves are very complex and it is difficult to locate points of maxima or minima, points of inflection or to determine the slope of the curve at a given point. If means are available for locating certain points on the curve, these problems may be solved by the approximate methods given here. Many expressions cannot be integrated exactly but definite integrals can be evaluated approximately using the trapezoidal rule or Simpson's rule. A more accurate method is presented here based on passing an arc of an nth degree parabola through $n+1$ points determined by a given function and determining the area under this parabola. This method may be extended to the estimation of areas in polar coordinates and to the estimation of the length of an arc of a curve in either rectangular or polar coordinates.

It is assumed that points determined by a given function can be obtained corresponding to values of an independent variable, x, which may be arranged in an arithmetical series. If the x variable is chosen as abscissa and if the corresponding ordinate in rectangular coordinates is denoted by y, such points may be represented by (x_i, y_i); $i = 0, 1, 2, \cdots$, etc. A simple transformation,

$$u \equiv \frac{x - x_0}{x_1 - x_0} = \frac{x - x_0}{\Delta x}, \tag{1}$$

changes the series of abscissae, x_i, into the particular series, i, in values of u. These values of u may be used to specify points if the curve is continuous and single-valued.

From the series of ordinates, y_i, a new series can be formed, namely:

$$(y_1 - y_0),\ (y_2 - y_1),\ (y_3 - y_2),\ \cdots,\ (y_n - y_{n-1}).$$

In the order given, the terms of this series may be denoted by the symbols, Δy_0, Δy_1, Δy_2, $\cdots$, Δy_{n-1}, and are designated as the first difference of y_0, the first difference of y_1, etc. In an analogous manner the series of second differences may be derived from the series of first differences. The series of second differences may be written as, $\Delta^2 y_0$, $\Delta^2 y_1$, $\Delta^2 y_2$, $\cdots$, $\Delta^2 y_{n-2}$. In the same manner this process may be

reapplied any number of times. The differencing operation may be briefly defined by the following identities:

$$\Delta^0 y_k \equiv y_k$$

and

$$\Delta^j y_k \equiv \Delta^{j-1} y_{k+1} - \Delta^{j-1} y_k.$$

$$j = 1,\ 2,\ 3,\ \cdots$$

$$k = 0,\ 1,\ 2,\ \cdots$$

In calculating these differences it is convenient to arrange the work in the following form:

$$\begin{array}{ccccccccc}
y_0 & & & & & & & & \\
 & \Delta y_0 & & & & & & & \\
y_1 & & \Delta^2 y_0 & & & & & & \\
 & \Delta y_1 & & \Delta^3 y_0 & & & & & \\
y_2 & & \Delta^2 y_1 & & \Delta^4 y_0 & & & & \\
 & \Delta y_2 & & \Delta^3 y_1 & & \cdot & \cdot & \cdot & \\
y_3 & & \Delta^2 y_2 & & \cdot & & & & \\
 & \Delta y_3 & & \cdot & & & & & \\
y_4 & & \cdot & & \cdot & & & & \\
 & \cdot & & \cdot & & & & & \\
\cdot & & \cdot & & & & & & \\
 & \cdot & & & & & & & \\
\cdot & & & & & & & &
\end{array}$$

Each difference is obtained from the two quantities immediately to the left of it by subtracting the upper quantity from the lower one.

Steffensen (1) has pointed out that if the problems of interpolation are to have definite solutions it is indispensible to have, beyond the numerical data, at least a certain general idea of the character of the function, $f(x)$, involved. In this discussion $f(x)$ means a real, single-valued function, continuous in a closed interval, say, $a \leq x \leq b$, and possessing in this interval a continuous derivative of the highest order of which use is made in deriving each formula under consideration.

Newton's formula for approximating a curve $y = f(x)$ passing through the points $(i,\ y_i)$; $i = 0,\ 1,\ 2,\ \cdots,\ n$ can be expressed in terms of the finite differences, $\Delta^i y_0$, as follows:

$$y = y_0 + \sum_{i=1}^{n} \left\{ \frac{\Delta^i y_0}{i!} \prod_{j=1}^{i} (u - j + 1) \right\} + R \qquad (2)$$

where $$R = \prod_{i=0}^{n}(u - i)\overline{\Delta x}^{\,n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

and by $f^{(n+1)}(\xi)$ is meant the $n + 1$ th derivative of $f(x)$ with respect to x; ξ being situated between the largest and the smallest of the numbers x, x_0, x_1, $\cdots$, x_n. This can be rearranged into the more useful form,

$$y = y_0 + \sum_{i=1}^{n} a_i u^i + R \tag{3}$$

where

$$a_i \equiv \sum_{j=i}^{n} \frac{C_{ij}}{j!}\Delta^j y_0, \qquad R = \sum_{i=1}^{n+1} C_{i(n+1)} u^i \frac{f^{(n+1)}(\xi)\overline{\Delta x}^{\,n+1}}{(n+1)!}$$

and C_{ij} can be calculated from the relation,

$$C_{ij} = C_{(i-1)(j-1)} - (j-1)C_{i(j-1)}$$

where $C_{ij} \equiv 1$ if $i = j$ and $C_{oj} \equiv 0$ if $j = 1, 2, 3, \cdots$.

Values of C_{ij} have been calculated for all values of i and j from 1 to 10 inclusive. These are enumerated in Table I:

TABLE I

Calculated Values of C_{ij}

j / i	1	2	3	4	5	6	7	8	9	10
1	+1	-1	+2	-6	+24	-120	+720	-5040	+40320	-362880
2	–	+1	-3	+11	-50	+274	-1764	+13068	-109584	+1026576
3	–	–	+1	-6	+35	-225	+1624	-13132	+118124	-1172700
4	–	–	–	+1	-10	+85	-735	+6769	-67284	+723680
5	–	–	–	–	+1	-15	+175	-1960	+22449	-269325
6	–	–	–	–	–	+1	-21	+322	-4536	+63273
7	–	–	–	–	–	–	+1	-28	+546	-9450
8	–	–	–	–	–	–	–	+1	-36	+870
9	–	–	–	–	–	–	–	–	+1	-45
10	–	–	–	–	–	–	–	–	–	+1

From Equations (1) and (3),

$$\frac{dy}{dx} = \frac{1}{\Delta x} \sum_{i=1}^{n} i a\, u^{i-1} + R' \tag{4}$$

$$\frac{d^2y}{dx^2} = \frac{1}{\Delta_x^2} \sum_{i=2}^{n} i(i-1)a_i u^{i-2} + R'' \tag{5}$$

where R' and R'' are given by the expressions

$$R' = \sum_{i=1}^{n+1} iC_{i(n+1)} u^{i-1} \frac{\Delta_x^n f^{(n+1)}(\xi)}{(n+1)!},$$

$$R'' = \sum_{i=2}^{n+1} i(i-1)C_{i(n+1)} \frac{u^{i-2}\Delta_x^{n-1} f^{(n+1)}(\xi)}{(n+1)!}. \tag{6}$$

Case I - Slope of $y = f(x)$

If four points, (x_i, y_i); $i = 0, 1, 2, 3$, are given satisfying $y = f(x)$ and equally spaced in the x direction ($x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \Delta x$) the point on the curve may be located where the slope is m approximately by the following method. From (4) $m\Delta x = a_1 + 2a_2u + 3a_3u^2 + R'\Delta x$ and by Table I

$$u = 1 - \frac{\Delta^2 y_0}{\Delta^3 y_0} \pm \sqrt{\left[\frac{\Delta^2 y_0}{\Delta^3 y_0}\right]^2 - \frac{\Delta^2 y_0}{\Delta^3 y_0} - 2\left[\frac{\Delta y_0}{\Delta^3 y_0}\right] + \frac{1}{3} + 2(m - R')\frac{\Delta x}{\Delta^3 y_0}} \tag{7}$$

and $$R' = (4u^3 - 18u^2 + 22u - 6)\frac{\Delta_x^3}{24} f^{(4)}(\xi).$$

It is suggested that R' be neglected to get the first approximation to u. x and y may be calculated from (1) and (2).

If $n + 1$ points of coordinates (x_i, y_i); $i = 0, 1, 2, \cdots, n$ are given satisfying $y = f(x)$ and equally spaced in the x direction, the slope D_p of the curve at the point, p, where $p = \frac{n}{2}$ and n is even may readily be approximated. From (3) and (4)

$$D_p = \frac{1}{\Delta x} \sum_{i=1}^{n} \sum_{j=i}^{n} i\left[\frac{n}{2}\right]^{i-1} \frac{C_{ij}}{j!} \Delta^j y_0 + R'$$

where

$$\Delta^j y_0 = y_j + (-1)jy_{j-1} + \frac{(-1)^2 j(j-1)}{2!} y_{j-2} + \cdots + (-1)^j y_0.$$

Then $$D_p = \frac{1}{\Delta x} \sum_{r=0}^{n} A_r y_r + R' \tag{8}$$

and
$$A_r = \sum_{i=1}^{n} \sum_{j=i}^{n} i\left(\frac{n}{2}\right)^{i-1} \frac{C_{ij}(-1)^{j+r}}{(j-r)!\,r!}. \tag{8}$$

One may define the differences

$$\begin{aligned}
\Delta' y_1 &\equiv y_{p+1} - y_{p-1},\\
\Delta' y_2 &\equiv y_{p+2} - y_{p-2},\\
&\;\;\vdots\\
\Delta' y &\equiv y_n - y_0
\end{aligned}$$

and assume that D_p can be expressed in terms of these differences as follows:

$$D_p = \frac{1}{\Delta x} \sum_{s=1}^{p} K_{sp} \Delta' y_s + R'$$

Then
$$\begin{aligned}
K_{pp} &= -A_0 &&= A_n\\
K_{(p-1)p} &= -A_1 &&= A_{n-1}\\
&\;\;\vdots\\
K_{1p} &= -A_{p-1} &&= A_{p+1}
\end{aligned}$$

In the above summations all terms involving the factorial of a negative number are to be ignored. A relation may be found between these constants by considering the curve, $y = D_p x$. Then $\Delta' y_s = 2s\Delta y_0$ and

$$D_p = 2\frac{\Delta y_0}{\Delta x} \sum_{s=1}^{p} sK_{sp} \quad \text{or} \quad \sum_{s=1}^{p} sK_{sp} = \frac{1}{2}.$$

These formulae may be applied to the following special cases:

(a) $$D_1 = \frac{1}{\Delta x}\left(\Delta y_0 + \frac{1}{2}\Delta^2 y_0\right) + R' = \frac{1}{\Delta x}\cdot\frac{\Delta' y_1}{2} + R'$$

(b) $$D_2 = \frac{1}{\Delta x}\left(\Delta y_0 + \frac{3}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{12}\Delta^4 y_0\right) + R' = \frac{1}{\Delta x}\left(\frac{2}{3}\Delta' y_1 - \frac{1}{12}\Delta' y_2\right) + R'$$

(c) $$D_3 = \frac{1}{\Delta x}\left(\Delta y_0 + \frac{5}{2}\Delta^2 y_0 + \frac{11}{6}\Delta^3 y_0 + \frac{1}{4}\Delta^4 y_0 - \frac{1}{20}\Delta^5 y_0 + \frac{1}{60}\Delta^6 y_0\right) + R'$$
$$= \frac{1}{\Delta x}\left(\frac{3}{4}\Delta' y_1 - \frac{3}{20}\Delta' y_2 + \frac{1}{60}\Delta' y_3\right) + R'. \tag{9}$$

Case II - Points of Inflection of $y = f(x)$

If four points (x_i, y_i); $i = 0, 1, 2, 3$ are given satisfying $y = f(x)$, and equally spaced in the x direction the position of a point of inflection may be estimated in this interval. From (5)

$$u = \frac{\Delta^3 y_0 - \Delta^2 y_0 - R''\overline{\Delta x}^2}{\Delta^3 y_0} \tag{10}$$

where
$$x = x_0 + u\Delta x$$

$$y = y_0 + (\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0)u + \frac{1}{2}(\Delta^2 y_0 - \Delta^3 y_0)u^2$$

$$+ \frac{1}{6}\Delta^3 y_0 u^3 + R$$

and
$$R = (-6u + 11u^2 - 6u^3 + u^4)\frac{f^{(4)}(\xi)\overline{\Delta x}^4}{4!}$$

$$R'' = (22 - 36u + 12u^2)\frac{f^{(4)}(\xi)\overline{\Delta x}^2}{4!}.$$

Case III - Areas and Lengths of Arc

From (1) and (3) the area under $y = f(x)$ may be evaluated approximately by the expression

$$A_0^n = \Delta x\{n(y_0 + R) + \sum_{i=1}^{n} \frac{a_i}{i+1} n^{i+1}\} \tag{11}$$

$$= \Delta x \sum_{r=0}^{n} C_r y_r + Rn\Delta x$$

where
$$C_r = n\sum_{i=1}^{n}\sum_{j=i}^{n}(-1)^{j+r}\frac{n^i C_{ij}}{(i+1)r!(j-r)!}, \quad r = 1, 2, \cdots, n$$

$$C_0 = n\left[1 + \sum_{i=1}^{n}\sum_{j=i}^{n}(-1)^j \frac{n^i C_{ij}}{(i+1)j!}\right].$$

It may be shown that in general $C_r = C_{n-r}$ and $\sum_{r=0}^{n} C_r = n$.

$$A_0^4 = \Delta x\{\frac{14}{45}(y_0 + y_4) + \frac{64}{45}(y_1 + y_3) + \frac{8}{15}y_2 + 4R\}$$

$$A_0^5 = \Delta x\{\frac{95}{288}(y_0 + y_5) + \frac{125}{96}(y_1 + y_4) + \frac{125}{144}(y_2 + y_3) + 5R\} \tag{12}$$

These equations may be adapted to deal with a curve in polar coordinates, $\rho = \phi(\theta)$, where

$$A_{p_0}^n = \frac{\Delta\theta}{2}\int_0^n \rho^2 du,$$

where in this case $u = \dfrac{\theta - \theta_0}{\Delta\theta}$ and

$$A_{p_0}^n = \frac{\Delta\theta}{2}\sum_{r=0}^{n} C_r \rho_r{}^2 + \frac{Rn}{2}\Delta\theta$$

and
$$R = \sum_{i=1}^{n+1} C_{i(n+1)} \frac{u^i [\phi^2(\xi)]^{(n+1)}}{(n+1)!} \overline{\Delta\theta}^{n+1}.$$

The length of arc corresponding to (12) is given by

$$S_0^n = \int_{x_0}^{x_n} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

or $S_0^n = \displaystyle\int_0^n \sqrt{\overline{\Delta x}^2 + (\sum_{p=1}^{n} i a_i u^{i-1} + R'\Delta x)^2}\, du$ approximately from (4).
Now this definite integral may be evaluated approximately according to the method of (11)

$$S_0^n = \sum_{r=0}^{n} C_r \sqrt{\overline{\Delta x}^2 + (\sum_{j=1}^{n} U_{jr}\Delta^j y_0 + R'\Delta x)^2} + Rn\Delta x \qquad (14)$$

and
$$U_{jr} = \sum_{i=1}^{j} \frac{iC_{ij} r^{i-1}}{j!}$$

R is on the function $\sqrt{1 + f'^2(x)}$, R' is on the function $f(x)$.
Values of U_{jr} have been calculated, see Table II.
This may be extended to polar coordinates giving

$$S_0^n = \sum_{r=0}^{n} C_r \sqrt{\rho_r{}^2\overline{\Delta\theta}^2 + (\sum_{j=1}^{n} U_{jr}\Delta^j \rho_0 + R'\Delta\theta)^2} + Rn\Delta\theta$$

where R' is on the function $\phi(\theta)$, R is on $\sqrt{\phi^2(\theta) + \phi'^2(\theta)}$.

TABLE II

Calculated Values of U_{jr}

j \ r	0	1	2	3	4	5
1	1	1	1	1	1	1
2	-1/2	1/2	3/2	5/2	7/2	9/2
3	1/3	-1/6	1/3	11/6	13/3	47/6
4	-1/4	1/12	-1/12	1/4	25/12	77/12
5	1/5	-1/20	1/30	-1/20	1/5	137/60

If the given values of a function are not exact, but contain errors of approximation and observation, these errors will affect the differences, and consequently the result of an interpolation (2). We can treat the argument of the function as though it is free of error and assume that all of the error is in the corresponding value of the function. The effect of such errors on the differences will be as if a difference table of the errors of the function is constructed. If the difference scheme of the errors is so prepared that all the other errors are zero, and only the error, e, to be considered is different from zero, the following scheme is obtained:

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0		0		0
	0		0	
0		0		e
	0		e	
0		e		$-4e$
	e		$-3e$	
e		$-2e$		$6e$
	$-e$		$3e$	
0		e		$-4e$
	0		e	
0		0		e
	0		0	
0		0		0

The effect of such an error always spreads out, and is greatest in the row in which the error is made. The effect of an error on a difference is determined by the magnitude of the binomial coefficients appearing in that column. This may serve as a means of estimating the error of calculation in a difference table.

In rounding off the values in a table, the error can amount to no more than one half unit in the last place. In the most unfavorable case, the following error table would be obtained. Here the errors are given in units of the last place:

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1/2		-2		8
	-1		4	
-1/2		2		-8
	1		-4	
1/2		-2		8
	-1		4	
-1/2		2		-8
	1		-4	
1/2		-2		8

The error of the rth difference is therefore at most $\pm 2^{r-1}$ units in the last place.

If, in a table prepared with approximated values of a function, the differences begin to oscillate irregularly about zero, the difference table should be terminated. One must then be content with an interpolation function of a corresponding order. In general, the total error of an interpolation function is equal to the error of the approximated value of this function due to: 1) error of measurement of y, and 2) poorness of fit of the approximating curve to the actual curve.

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APPLICATION OF THE SUMMATION BY PARTS FORMULA TO SUMMABILITY OF SERIES

Tomlinson Fort

1. Most studies on summability have been concerned with general theories and but little attention has been given to elementary methods for testing numerical series for summability. The present paper is written to show how the summation by parts formula can be used in testing numerical series.

2. *The Fundamental Formula*

The summation by parts formula is

$$\sum_{i=m}^{n} \epsilon_i s_i = \epsilon_{n+1} \sum_{i=m}^{n} s_i - \sum_{i=m}^{n} \Delta\epsilon_i \sum_{j=m}^{i} s_j. \tag{1}$$

where $\Delta\epsilon_i = \epsilon_{i+1} - \epsilon_i$

This formula is an algebraic identity. It has been known for a long time and is sometimes called the Abel summation by parts formula. It is to be noted that ϵ_{n+1} does not occur in the left-hand member, and, so far as the identity itself is concerned, can be considered as arbitrary.

The formula serves to "separate" the s_i from the ϵ_i and it is on this that its applications usually depend.

3. *The Arithmetic Mean*

In this section we give two theorems relative to the arithmetic mean of the first order. The reader will find that he can state and prove other theorems.

THEOREM I. *Hypotheses:* (a) $\epsilon_n \to 0$ $\quad(b)$ $\frac{1}{n}\sum_{i=1}^{n} i|\Delta\epsilon_i| \to 0$

(c) $\left|\frac{1}{n}\sum_{i=1}^{n} s_i\right| < H$

Conclusion: $\frac{1}{n}\sum_{i=1}^{n} \epsilon_i s_i \to 0$

THEOREM II. *Hypotheses:* (a) $\epsilon_{n+1}\frac{1}{n}\sum_{i=1}^{n} s_i \to s$ $\quad(b)$ $\frac{1}{n}\sum_{i=1}^{n} s_i \to 0$

(c) $\frac{1}{n}\sum_{i=1}^{n} i|\Delta\epsilon_i| < K$

Conclusion: $\frac{1}{n}\sum_{i=1}^{n} \epsilon_i s_i \to s.$

In order to prove the first theorem we turn to formula (1) and write

$$(4)\qquad \left|\frac{1}{n}\sum_{i=1}^{n}\epsilon_i s_i\right| \leqq \left|\epsilon_{n+1}\right|\left|\frac{1}{n}\sum_{i=1}^{n}s_i\right| + \frac{1}{n}\sum_{i=1}^{n} i\left|\Delta\epsilon_i\right|\left|\frac{1}{i}\sum_{j=1}^{i}s_j\right|$$

$$\leqq \left|\epsilon_{n+1}\right| H + H\frac{1}{n}\sum_{i=1}^{n} i\left|\Delta\epsilon_i\right|$$

The theorem follows immediately.

In order to prove theorem II we fix our attention on

$$\frac{1}{n}\sum_{i=1}^{n} i\,\Delta\epsilon_i\,\frac{1}{i}\sum_{j=1}^{n} s_j$$

We shall show that under the hypotheses of the theorem this approaches zero. The theorem will then be immediate. Let an $\epsilon > 0$ be given and choose N so that $\frac{1}{i}\left|\sum_{j=1}^{i} s_j\right| < \frac{\epsilon}{2K}$ whenever $i > N$. Now hold N fast and choose P so large that

$$\frac{1}{n}\sum_{i=1}^{N} i\left|\Delta\epsilon_i\right|\left|\frac{1}{i}\sum_{j=1}^{i} s_j\right| < \frac{\epsilon}{2}$$

when $n > P$.

Under these circumstances

$$\left|\frac{1}{n}\sum_{i=1}^{n} i\,\Delta\epsilon_i\,\frac{1}{i}\sum_{j=1}^{i} s_i\right| \leqq \frac{1}{n}\sum_{i=1}^{N} i\left|\Delta\epsilon_i\right|\left|\frac{1}{i}\sum_{j=1}^{i} s_j\right|$$

$$+\frac{1}{n}\sum_{i=N+1}^{n} i\left|\Delta\epsilon_i\right|\left|\frac{1}{i}\sum_{j=1}^{i} s_i\right| \leqq \frac{\epsilon}{2} + \frac{\epsilon}{2K}\frac{1}{n}\sum_{i=1}^{n} i\left|\Delta\epsilon_i\right| < \epsilon.$$

We shall now give two numerical examples to illustrate the use of these theorems.

Consider the summability of the sequence $(-1)^n\sqrt{n}$ by the method of the arithmetic mean of first order. Let $s_n = (-1)^n n$ and $\epsilon_n = \frac{1}{\sqrt{n}}$ Then

$$(5)\qquad \frac{1}{n}\sum_{i=1}^{n} s_i = \frac{1}{n}\sum_{i=1}^{n}(-1)^i i = \begin{cases} \tfrac{1}{2} \text{ if } n \text{ is even.} \\ -\tfrac{1}{2}\,\dfrac{n+1}{n} \text{ if } \quad \text{is odd.}\end{cases}$$

This can be proved by the formula for the sum of an arithmetic progression. (Sum the positive and the negative terms separately.) From (5) we note that $\frac{1}{n}\sum_{i=1}^{n} s_i$ is bounded. Moreover $\epsilon_{n+1} \to 0$. Also

$$(6)\qquad \frac{1}{n}\sum_{i=1}^{n} i\left|\Delta\epsilon\ \right| = \frac{1}{n}\sum_{i=1}^{n} i\left|\frac{1}{\sqrt{i+1}} - \frac{1}{\sqrt{i}}\right| = \frac{1}{n}\sum_{i=1}^{n} i\,\frac{\sqrt{i+1}-\sqrt{i}}{\sqrt{i+1}\,\sqrt{i}}$$

$$< \frac{1}{n}\sum_{i=1}^{n}(\sqrt{i+1}-\sqrt{i}) = \frac{1}{n}(\sqrt{n+1}-1) \to 0.$$

It follows from Theorem I that $(-1)^n\sqrt{n}$ is summable to zero by the method of the arithmetic mean of order 1. This is a result that can be proved by other methods but which is not immediately evident. We notice in particular that the partial sum namely $\frac{1}{n}\sum_{i=1}^{n}(-1)^i\sqrt{i}$, is of the order of $\frac{1}{\sqrt{n}}$.

Next we consider the sequence $(-1)^n\sqrt{n}\log n$. Let $s_n = (-1)^n\sqrt{n}$ and $\epsilon_n = \log n$. Now we have just proved that $\frac{1}{n}\sum_{i=1}^{n} s_i \to 0$. Moreover,

$$\epsilon_{n+1}\,\frac{1}{n}\sum_{i=1}^{n} s_i = (\log(n+1))\frac{1}{n}\sum_{i=1}^{n}(-1)^i\sqrt{i} \to 0$$

This is true, since $\frac{1}{n}\sum_{i=1}^{n}(-1)^i\sqrt{i}$ approaches zero of the order of $\frac{1}{\sqrt{n}}$. Moreover

$$\frac{1}{n}\sum_{i=1}^{n} i\left|\Delta\epsilon_i\right| = \frac{1}{n}\sum_{i=1}^{n} i(\log(i+1) - \log i) = \frac{1}{n}\sum_{i=1}^{n} i\cdot\frac{1}{\xi_i}$$

where $i < \xi_i < i+1$. Consequently

$$\frac{1}{n}\sum_{i=1}^{n} i\left|\Delta\epsilon_i\right| = \frac{1}{n}\sum_{i=1}^{n}\frac{i}{\xi_i} < \frac{1}{n}\sum_{i=1}^{n} 1 = 1.$$

Hence, in this example $\frac{1}{n}\sum_{i=1}^{n} i\left|\Delta\epsilon_i\right|$ is bounded. We conclude by Theorem II that $(-1)^n\sqrt{n}\log n$ is summable to zero by the method of the arithmetic mean of order 1.

4. *Repeated Summation by Parts*

If formula (1) is applied to the sum $\sum_{i=1}^{n}\Delta\epsilon_i\sum_{j=1}^{i} s_i$ in (1) with $\Delta\epsilon_i$ replacing the ϵ_i in the formula and $\sum_{j=1}^{i} s_i$ replacing s_i we have

$$(7)\qquad \sum_{i=1}^{n}\epsilon_i s_i = \epsilon_{n+1}\sum_{i=1}^{n} s_i - (\Delta\epsilon_{n+1})\sum_{i=1}^{n}\sum_{j=1}^{i} s_j + \sum_{i=1}^{n}\Delta^2\epsilon_i\sum_{j=1}^{i}\sum_{k=1}^{j} s_k.$$

Here $\Delta^2\epsilon_i = \Delta(\Delta\epsilon_i)$. Similarly $\Delta^n\epsilon_i = \Delta(\Delta^{n-1}\epsilon_i)$. If we now apply (1) to the last sum appearing in (7) and repeat we get the following formula,

$$(8) \qquad \sum_{i=1}^{n}\epsilon_i s_i = \epsilon_{n+1}\sum_{i_1=1}^{n}s_{i_1} - \Delta\epsilon_{n+1}\sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}s_{i_2} +$$

$$\Delta^2\epsilon_{n+1}\sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}s_{i_3} + \cdots + (-1)^p\Delta^p\epsilon_{n+1}\sum_{i_1=1}^{n}\sum_{i_2=1}^{i_p}\cdots$$

$$\sum_{i_{p+1}=1}^{i_p}s_{i_{p+1}} + (-1)^{p+1}\sum_{i_1=1}^{n}\Delta^{p+1}\epsilon_{i_1}\sum_{i_2=1}^{i_2}\cdots\sum_{i_{p+2}}^{i_{p+1}}s_{i_{p+2}}.$$

Formula (7) or for that matter formula (8) may be helpful when applied to summability. For example if each term on the right approaches a limit then so does the left-hand member.

5. *General Summability by Triangular Matrix*

Many special applications can be made of formulas (1), (7), or (8). For example if we write

$$\sum_{i=1}^{n}c_{in}s_i = \sum_{i=1}^{n}a_{in}\sigma_i b_{in}$$

and let $a_{in}\sigma_i$ replace s_i in (7) and b_{in} replace ϵ_i we have the following result. Variations of the theorem will be apparent.

THEOREM III. Hypotheses: (a) $b_{n+1,n} \to 0$ (b) $\left|\sum_{i=1}^{n}a_{in}\sigma_i\right| < H$

(c) $n\,\Delta b_{n+1,n} \to 0$ (d) $\sum_{i=1}^{n}i\Delta^2 b_{in} \to 0$

Conclusion: $\sum_{i=1}^{n}c_{in}s_i \to 0$

Proof: From (7)

$$\left|\sum_{i=1}^{n}c_{in}s_i\right| \leqq |b_{n+1,n}|H + H|\Delta b_{n+1,n}|n + H\sum_{i=1}^{n}i|\Delta^2 b_{in}|$$

The theorem follows.

In case $b_{in} = \epsilon_i$ we write

$$\sum_{i=1}^{n}c_{in}s_i = \sum_{i=1}^{n}c_{in}\sigma_i\epsilon_i.$$

Theorem III may be applied to this form.

6. *Repeated Sums*

This paper was introduced by a discussion of summability by the method of the arithmetic mean of order 1. If we are given an infinite series $\sum_{i=1}^{\infty} u_i$ then the sum by the method of the arithmetic mean is $\lim_{n\to\infty} S_n$, where

$$S_n = \frac{1}{n}\sum_{i=1}^{n} s_i = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} u_j .$$

Now if the terms of the series are factored thus $u_j = a_j \epsilon_j$ we have

$$S_n = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} a_j \epsilon_j$$

Summation by parts done twice permits us to "separate" the ϵ_j from the a_j. Thus:

$$S_n = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} a_j\epsilon_j = \frac{1}{n}\sum_{i=1}^{n}[\epsilon_{i+1}\sum_{j=1}^{i} a_j - \sum_{j=1}^{i}\Delta\epsilon_j \sum_{k=1}^{j} a_k]$$

$$= \epsilon_{n+2}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} a_j - \frac{2}{n}\sum_{i=1}^{n} i(\Delta\epsilon_{i+1})\frac{1}{i}\sum_{j=1}^{i}\sum_{k=1}^{j} a_k$$

$$+ \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} j(\Delta^2\epsilon_j)\frac{1}{j}\sum_{k=1}^{j}\sum_{m=1}^{k} a_m .$$

From this formula we can draw a variety of conclusions such as the following theorems.

THEOREM IV. *Hypotheses:* (a) $\left|\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} a_j\right| < H$. ($b$) $\epsilon_n \to 0$. (c) $\frac{1}{n}\sum_{i=1}^{n} i|\Delta\epsilon_{i+1}| \to 0$. ($d$) $\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} j|\Delta^2\epsilon_j| \to 0$.

Conclusion:

$\sum_{i=1}^{\infty}\epsilon_i a_i$ *is summable to zero by the method of the arithmetic mean.*

THEOREM V. Hypotheses: (a) ϵ_n approaches a limit (b) $\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} a_i$ *approaches a limit* (c) $i\Delta\epsilon_{i+1}$ *and* $j\Delta^2\epsilon_j$ *define regular methods of summability.*

Conclusion: $\sum_{i=1}^{\infty}\epsilon_i a_i$ *is summable by the method of the arithmetic mean.*

The Cesarò mean of integral order, r, is given by the following

formula (See Fort, Infinite Series, page 204).

$$C = \lim_{n\to\infty} C_{nr}$$

$$C_{nr} = \frac{r!}{(n+1)(n+2)\cdots(n+r)} \sum_{i_1=0}^{n} \sum_{i_2=0}^{i_1} \cdots \sum_{i_r=0}^{i_{r-1}} s_{i_r}$$

If $s_n = \tau_n \cdot \epsilon_n$ we write

$$C_{nr} = \frac{r!}{(n+1)(n+2)\cdots(n+r)} \sum_{i_1=0}^{n} \sum_{i_2=0}^{i_1} \cdots \epsilon \sum_{i_r=0}^{i_{r-1}} \tau_{i_r} \epsilon_{i_r}$$

Repeated summation by parts will serve to separate the τ and the ϵ and a variety of conclusions can be drawn which can not be drawn from the discussion under section 5. Summation by parts is simple, the coefficients following the binomial law. The formula is written out for the case of $r = 3$ only and one conclusion drawn. The reader can state other theorems.

Let $$D_{n3} = \frac{(n+1)(n+2)(n+3)}{3!}$$ and

$$\overline{C}_{n3} = \frac{1}{D_{n3}} \sum_{i=1}^{n} \sum_{j=0}^{i} \sum_{k=0}^{j} \tau_k .$$

Then

$$C_{n3} = \epsilon_{n+3}\overline{C}_{n3} - 3\frac{1}{D_{n3}} \sum_{i=0}^{n} D_{i3}(\Delta\epsilon_{i+2})\overline{C}_{n3}$$

$$+ 3\frac{1}{D_{n3}} \sum_{i=0}^{n} \sum_{j=0}^{i} D_{j3}(\Delta^2 \epsilon_{j+1})\overline{C}_{j3}$$

$$- \frac{1}{D_{n3}} \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{k=0}^{j} D_{k3}(\Delta^3\epsilon_k)\overline{C}_{k3}$$

THEOREM VI. Hypotheses: (*a*) $\epsilon_n \to$ a limit. (*b*) $D_{i3}(\Delta\epsilon_{i+2})$ is bounded (*c*) $D_{j3}(\Delta^2\epsilon_{i+1})$ is bounded (*d*) $D_{k3}\Delta^3\epsilon_k \to$ a limit (*e*) $\overline{C}_{n3} \to$ a limit.

Conclusion: $C_{n3} \to$ a limit.

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THE PERSONAL SIDE OF MATHEMATICS

This department desires especially articles showing what mathematics means to people in various professions and historical articles showing what classic mathematics meant to those who developed it. Material intended for this Department should be sent to the Mathematics Magazine, 14068 Van Nuys Blvd., Pacoima, California.

MATHEMATICS AND HISTORIOGRAPHY

Oliver E. Glenn

It is a part of more recent philosophy by Benedetto Croce that all genuine knowledge is historical knowledge, and that mathematics, even, does not present any essential exception to this principle. Life and reality are history and history alone, according to Croce, and since a main objective of the mathematician, as well as a major element in mathematics, is the interpretation of reality, mathematics recognizes the proposition that the presentation of its substance in print is a phase of historicism, (Science of history).

Consequently we must go farther with Croce in his critical development of the principles of historiography. There are at least two main principles to be followed by a writer of a piece of history. He not only must have a proper subject to write about; he should pose and develop a definite historical problem. Some existing histories are histories without the historical problem, as are many of the writings of Von Ranke, of which Croce gives a notable critique in *History as the Story of Liberty*, (1941). The *Geschichte der Renaissance in Italien*, (1890), by Jakob Burckhardt, is an example of a history without the historical problem. One can read the latter work with great absorbtion and enjoyment and, when through, be troubled very much by the question of just what has been attained by the reading; what problem has been sounded. One will then read the work again with enjoyment perhaps greater but with the same result, that the reading leads to no historical objective because the writer has posed no historical problem. Modern historians have made much of the so-called scientific attitude and have said that the facts of history, (What has happened), if well presented, are history, which will stand or fall according as the alleged facts are, or are not, facts. But, in writing of Robespierre, to cite an instance, one can recite the facts of his career meticulously without achieving a very close contact with the essential and true history of Robespierre and of the movements in which he figured. What will be achieved will be a chronicle, which is not a history. A true historical problem would deal with both what Robespierre did and why he acted as he did, and how both his acts and his intentions affected

France's revolutionary struggle for liberty.

A second criterion for the writing of history is that the historian should not shirk or belittle the passing of historical and moral judgment of historical movements and of men. To give a chronicle of events without passing historical judgment, especially from the standpoint of the relations of those facts with freedom's progress, but also of their relations with other issues related to the humane, is to assume that an event may be historically ascertained while judgment upon it is reserved, but this assumption is nearly always false.

Now when we write new mathematics, as historical knowledge, what takes the place of the statement of the historical problem, in historicism as it is rightly understood? A new mathematical conclusion is, so to say, a link in a hereditary chain of ideas. Its author, conscious of the findings of his predecessors in the field, makes a new combination of ideas, and from this combination springs a novel viewpoint, succession of ideas, logical origination, and organization. He states his mathematical problem when he makes clear the nature of his own logical system in comparison with what previously was known. In reply to a possible objection that the study of a natural phenomenon, a problem of reality, external as compared with the internal special realities of pure mathematics itself, may lead to mathematics that is surprisingly new, but the situations which are affirmed by such a discovery are nevertheless historical. Its identification as a historical fact, in retrospect and in anticipation, is what we mean by posing the mathematical problem.

The writer of new mathematics passes mathematical judgment, in the sense of historicism, when he contends for the essential novelty, significance and promise of his new system; secondly, when he states whether his new theory cuts under and so tends to render less important any of the theories related in the environment, and whether his doctrine generalizes any related doctrine, or is itself capable of significant generalizations, and whether his system evidently lends itself to effective new interpretaions of phases of external reality.

We remark here that the process of generalization of a mathematical formulary, or principle, does not necessarily render it less important, in the science, than it was. The Galois Field, for example, has been generalized to a complex domain with respect to an arbitrary composite modulus, but without changing the status of the Galois Field.

So an author states a problem and develops it to a point where judgments can be passed, and, correspondingly, some criteria for effective oral and printed mathematical publicity can be stated. To begin with the work of the small-problem solvers, who are still with us in various journals, and refuse to have their enthusiasms suppressed,—what problems of historiography are they dealing with? There are two at least. The time has not yet come when a new mathematical beginning, upon which can be based an induction, or a generalization, cannot be found in the every-day instructional miscellanea of example work, and

in other work with the small-problem. In fact such opportunities increase in number with the modern increase in the number of new theories. Credit for previous discoveries at this level should be given also by writers. Secondly, new theories require correlation with the relevant external and internal realities in a sense more thorough-going than the time of the originators of those theories permits. It is not always a yeoman's task to amplify a new mathematical system by work in the field of the small problem. Laplace's *Mécanique Céleste,* the work that was translated by Bowditch, is, from one point of view, a study of the relations of the Differential and Integral Calculus to a broad problem of external reality. Other work of this general nature, for the Calculus and for other systems, includes that of the mathematicians S. A. Corey, R. D. Carmichael, DeLand, Dickson, Finkel, Rev. Hawkesworth, W. J. Greenstreet, Greenwood, Holmes, Artemus Martin, Safford, Scheffer, and G. B. M. Zerr.

An editor might do well to indicate what system has been amplified, in a small way or large, when a solution of any small-problem is being published.

Our next special emphasis will be on the mathematical abstract. Such publications are necessarily in the form of chronicle only. They have that status in historicism. They are primarily designed to give, for the benefit of a few who may be working in a closely related field, a chronicle of what has been accomplished, without proofs, or any adequate statement of either a problem or a judgment, in the sense of historicism. I do not think that new mathematics should ever be published in the form of such an abstract. A mathematical development, with proofs and all included, is already the ultimate of condensation of an intellectual product. To give it its first publication merely in abstract, is, - to use phraseology due to Socrates, - a sort of outrage on the understanding. It is different with the mathematical review of work which has been published in full, and with those mathematical histories in which the review technique is used. The great intellectual traditions to be observed in the mathematical world; scientific conscience, freedom of expression, and intellectual and therefore humane progress, require that original production should concern itself, not only with summaries, but with details and actual proofs. Nature, reality, has a great propensity for going into detail.

Closely related to the abstract is the program of the mathematical convention. Speakers there should have time in which to develop their subjects. A speaker will speak to his whole audience, and not merely to a few, if he presents; (a) his problem and conclusions for the benefit of specialists in the field of his problem, (b) his problem in the light of historiography, (c) his mathematical judgments.

The trend of this discussion also raises some questions about the practices of recent Congresses which have been satisfied to publish the papers read at their meetings, only in abstract, or so briefly that

essential proofs had to be omitted. A Congressist may be erudite, but, in the direction represented by his communication, a future generation of workers will likely be less erudite. The discourses without proofs will then very probably rest unread upon library shelves; examples of chronicle historiography. One could hardly protest too emphatically against a tendency to create any area of transcendentalism in mathematical science; in particular against this stilted form of publication where the proof; the real meat of the fare, is left out. In extreme cases such publication might prove to be mere concealment in relation to research which had not reached its true objectives. It is one of the achievments of the thought of the twentieth century that we have come a long way from transcendentalism and illuminism, which should never have vexed American thought. It is well known that, during the period when the world at large was enjoying the greatest measure of freedom ever attained, (the Victorian era), New England had passed from unquestioned intellectual leadership to a state of stagnation, and largely as an aftermath of transcendentalism. The exceptions were Agassiz, Howells, Pierce, and a few others, who were effective only in so far as they got away from transcendentalism, or as they were never really subject to it. This is said with a knowledge of that rejoinder which is often made, - Do you then not believe in idealism?, - as if the ideal were something inseparable from the obscure and the non-reasonable, and separable from life as it is lived.

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DON'T CALL IT SCIENCE

James E. Foster

"Mathematics is the science in which we never know what we are talking about nor whether what we say is true".

— Bertrand Russell

On the basis of form, non-mathematicians should have picked up the Russell definition and hurled it into the teeth of the mathematical world. The mathematician's smug self-assurance, his either-it-is-or-it-isn't attitude, coupled with the drudgery traditionally associated with elementary mathematics, have planted him on a pedestal of sorts, and to be able to pellet him up there with the words of one of his distinguished colleagues . . . .

The obvious explanation of the non-existence of such a devil-quoting-scripture situation is that Russell came up with his epigram as a mathematician writing about mathematics and thereby assured its being kept secret from the world at large. The explanation would probably bd valid, were the author any one but Russell, whose writings have an audience far beyond the mathematical world and have made numerous non-mathematicians sample his less technical mathematical work. Furthermore, the definition has been quoted frequently enough for any alert layman looking for missiles to be aware of it.

Possibly grammar school arithmetic so frightens children that after they grow up they are afraid to fight back, even with ammunition supplied by the enemy. Or, maybe they suspect some subtle trickery in it all and conclude the smart thing is to have nothing to do with it.

Had some layman with a de-bunking urge quoted Russell to a mathematician, the mathematician would probably have replied: "Of course, you're taking it out of its context .... Russell was still pretty young when he wrote that, and over-impressed by some epigrams he had heard in an Oscar Wilde play ..... Besides, he had his tongue in cheek ....."

Actually, however, while mathematicians may concede the definition suggests a painful effort at being clever, they do not dismiss it on that account. Instead, they consider it with all the seriousness they would give to an assortment of x's and y's strung together into an equation.

The fact that a flippant definition such as this might be concocted in the twentieth century and taken at face value is in itself a commentary on mathematics. Historically, mathematics is among the oldest fields of intellectual effort. Much of mathematics currently taught below the college level antedates Christianity. Yet, for all its antiquity, an epigrammatic definition of it is taken seriously, even

considered profound.

One finds no such situation among the sciences. The biologist, the chemist, the physicist are consistently and humorlessly matter of fact when they define their respective fields. So, too, is the practitioner in any of the newer sciences.

Unlike the mathematician, they are ready, even eager, to sally into definition. Pick up a text book, any text book, devoted to any science, and you will find a definition in chapter one, probably in paragraph one. It is true that working definitions of their respective fields are found in grammar school arithmetic texts and high school geometry texts, the latter explaining that the word "geometry" comes from two Greek words and literally translates into "earth measurement". But, where does one find a definition of mathematics as such, a definition broad enough to include its various domains? Not until one has advanced through, or at least sampled, those domains.

Here again mathematics differs from the sciences. The student making his initial acquaintance with a science learns of its more general aspects and after that specializes. Not so in mathematics. Acquaintance is made through arithmetic, a highly specialized field, certainly less general than algebra, which follows it in the conventional course of study. Not until one gets into such fields as mathematical philosophy or the algebra of sets is mathematics as a unified domain ordinarily considered.

The situation provokes inquiry. Are the various fields of mathematics independent domains of study having nothing more in common than the fact they are collectively referred to as "mathematics" or is mathematics a subject that can be defined more precisely than by enumerating its branches? If it is, why are not its broad outlines presented to beginning students? And, why the difficulty at definition?

The difficulty can be laid at the door of mathematicians, who whould know better. Such a statement may be an impertinence, coming as it does from a mere layman with slight training in and superficial familiarity with mathematics and whose interest in it is that of a dabbler. Nevertheless, it stands, and will be, it is hoped, adequately substantiated for present purposes.

There have been a number of definitions of mathematics. Russell, when not in epigramatic mood, referred to it as "the class of all propositions of the form 'A implies B', where A and B are themselves propositions involving the same variables but not constants except logical constants". Benjamin Pierce defined it as "the science which draws necessary conclusions", and Whitehead called it "the science concerned with the logical deduction of consequences from the general of all reasoning".

Though these definitions vary, implication and abstraction are implied in all of them. These are the essential elements in all mathematics. Thus, the numbers in arithmetic are themselves abstractions.

An arithmetic equation (e.g. 2 + 3 = 5) exists independently of any items to which the numbers might apply (apples, heart beats, or days). The operation expressed by the equation is an implication of certain assumptions which include the sequence of the number system and one or more of the basic postulates of algebra.

Mathematics might, therefore, be defined as the domain of abstract implication. This definition, it will be noted, is consistent with those given above. While it includes quantitative and spatial concepts, it is not limited to them. Thus, symbolic logic is a branch of mathematics, although it does not utilize either type of concept.

The persistence with which mathematics is referred to as a science must cause confusion as to its nature. In a time when science is virtually deified by both the intellectual and the man on the street, it is perhaps natural that mathematicians would have their field known as a science, just as physicists during the eighteenth century would refer to the laboratory gadgets of the time as "philosophical machines" in deference to the respect philosophy then commanded. Eighteenth century physics, however, belonged to science, and twentieth century mathematics is in the field of philosophy.

Nevertheless, mathematics is almost routinely referred to as a science. Webster's unabridged dictionary offers three definitions, each beginning "The science..." Even the mathematicians who emphasize the essentially abstract nature of their field (Pierce, Russell, Whitehead among others) call it science.

Not only is mathematics considered a science, it is even further categorized by educators as a physical science. In the typical liberal arts college, the undergraduate may submit credits in mathematics to satisfy requirements in physical science, but not in philosophy. It is apparently assumed that mathematics is a physical science because advanced study in chemistry, physics, and astronomy requires something more than average exposure to it. By the same line of reasoning, it might with equal validity be contended that mathematics is a social science, since social research so frequently involves statistical techniques. The fact is that mathematics is neither a physical nor a social science, if for no other reason than that it is not a science.

Essentially, a statement in mathematics is of the form: If *a*, then *b*. Its concern is limited to the implications of certain assumptions, which are consistent with one another. The raw material of science, however, is accumulated experience. Phenomena are observed as they occur in nature or under laboratory conditions, and on the basis of these observations causative processes are described. Typically, such descriptions are tentative, and are discarded, modified, or confirmed, as evidence is amassed.

Though the scientist may use mathematics in interpreting his observations and the observations of the mathematician may suggest basic assumptions from which to reason, the two fields are essentially

different. Science is inductive, mathematics is deductive. In this connection, Poincare suggested that mathematics shared an easement with science through the process of mathematical induction. (See "Science and Hypothesis" in *The Foundations of Science*, New York, 1913, pp. 31ff). It is true that mathematical induction involves the use of an hypothesis the validity of which is not immediately evident from the basic assumptions of mathematics. Essentially, however, mathematical induction is a deductive process through which the hypothesis (an assumed algebraic relationship) is derived from its arithmetic occurrence in a single case. Neither in mathematical induction nor in any other form of mathematics is there the piling up of evidence found in science. Actually, there are generalizations in mathematics for which there are no known exceptions, but which are not accepted as valid because the mere amassing of evidence is not involved in mathematical proof. As an illustration, the proposition $a^n + b^n = c^n$ has never been satisfied by rational numbers where n is an integer greater than 2, although exhaustive attempts to find rational solutions with such integers have been made. The consistent failure is good scientific evidence that such solutions do not exist. The mathematician, however, will do no more than say such solutions probably do not exist, but, after all, there is no mathematical evidence to justify such an assertion. On the other hand, the proposition of Pythagorus is proved without measuring the sides of a single right triangle, and were some one to check the validity of that proposition by measuring the sides of a right triangle and to find the sum of the squares of the base and the altitude to be greater or less than the square of the hypotenuse, no right-thinking mathematician would be disturbed. Instead, he would attribute the difference to errors in measurement or in computation, to the presumed right angle being something more or less than 90 degrees, or to the non-existence of Euclidean straight lines in the particular triangle. With a self-assurance no scientist would dare assume, he would insist the proposition of Pythagorus to be true in a Euclidean universe, and, if measurements seemed to disprove it, either the measurements were faulty or the conditions under which the triangle existed were not Euclidean. And, were the scientist to attempt further verification by measuring the sides of large numbers of triangles and computing averages, the mathematician would tell him all that might be well and good scientifically, but mathematically it was a waste of time.

But, with curriculum designers cataloging mathematics with the physical sciences and mathematicians referring to their field as a science, people in general naturally think of it as one. Superficially, this may seem a fortunate misapprehension in an age in which science is deified. In the long run, however, it is handicapping to mathematics. Quackery may flourish through being misunderstood; the same, however, cannot be said for any valid field of intellectual activity.

On the contrary, the growth of many of the sciences has been handicapped by illusions as to their natures. Progress in chemistry, for example, has been to a large extent the result of counteracting belief in alchemy.

In a way, cataloging mathematics as a science is even more serious than confusing chemistry with alchemy, since the former classification is accepted by people who should know better, while the latter is not. After all, no one with even a smattering knowledge of chemistry takes alchemy seriously. Mathematicians, however, frequently, if not generally, think of their field as a science. Naturally enough, so do other people, and when it dawns on them that its methods are essentially different from those of such varied sciences as physics, geology, and biology, they may come to realize some one did a bum job of classifying. The chances are, however, that they will decide that mathematics is some sort of auxiliary science, not a top flight science, but rather an incidental one, not to be taken too seriously, but of use when problems are to be solved.

Now, handy men are useful members of society, but they are not what ambitious boys hope to be when they grow up. Serious scholarship is not directed into fields having no other justification than their incidental utility in other fields. There can be a valid and a satisfying interest in mathematics for its own sake, but that interest is impossible if mathematics is conceived as nothing more than problem solving.

This is not to deny its usefulness or to deplore that usefulness. But, there is more to it than that. While the techniques of problem solving can never be as important as the problems, the principles involved in those techniques, the pure reason implicit in all mathematics can furnish a form of intellectual stimulation that transcends the manipulations involved in getting the answers.

To realize this, one must appreciate the basic difference between mathematics and science in any forms either may take. Without such an appreciation, one cannot comprehend the stature of mathematics; instead, one must regard it simply as a means to accomplishment in some scientific discipline, rather than as the continuously stimulating and satisfying intellectual activity it is.

The first step in creating such an appreciation must be the development of an understanding of what mathematics really is. To call it a science is to give an immediate misimpression of it. A definition beginning "the science ...," may be correct in everything after those first two words. Those two words, however, can cause plenty of damage, since they automatically assign mathematics to the area of codified observation rather than to that of abstraction and implication. Yes, the scientific bandwagon is a pleasant place these days, with every one on it regarded with something of awe. But, mathematics can never be comfortable where it does not belong. It will go a lot further,

if it describes itself for what it is, as a field of intellectual activity that does not belong to science, if for no other reason than that it transcends science. Agree that it is indispensible to science, but add that it can be, and is to every one who has pursued it, a source of interest for its own sake.

822 Colfax Street
Evanston, Illinois.

[Not a few mathematicians will differ vigorously with Mr. Foster's viewpoint, and we hope some one of these will send us his viewpoint,

Ed.]

PROBLEMS AND QUESTIONS

Edited by

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction. Readers are invited to offer heuristic discussions in addition to formal solutions.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

161. *Proposed by E. P. Starke, Rutgers University.*

Show that there are infinitely many sets of three integers whose product equals the sum of their squares.

162. *Proposed by Howard Eves, State University of New York.*

A variable triangle $A'B'C'$ of constant area is inscribed in a triangle ABC, so that A' lies on BC, B' on CA, and C' on AB. If AA', BB', CC' are divided by A'', B'', C'', respectively, in the same ratio, show that the triangle $A''B''C''$ has a constant area.

163. *Proposed by P. A. Piza, San Juan, Puerto Rico.*

(1) Find positive integers A, B such that $(A + B)^2 = 10^6 A + B$. (2) Find positive integers C, D, n such that $(C + D)^{2n} = 10^6 C + D$.

164. *Proposed by J. M. Howell, Los Angeles City College.*

There are four doors and four keys, each of which fits one and only one door. What are the probabilities that all of the doors will be opened in exactly k trials ($k = 4, 5, \cdots, 10$), where the trying of a key in a lock is considered a trial.

165. *Proposed by Leon Bankoff, Los Angeles, Calif.*

A line segment is divided into two parts, a and $k - a$. On each of these parts as a diameter a semicircle is drawn. Find the locus of the midpoint of the line composed of the arcs of the semicircles.

166. *Proposed by W. B. Carver, Cornell University.*

What is the maximum length of a line segment that can be drawn in the smaller segment of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ cut off by the line $x = h$, $a > b$, $0 \leq h < a$?

167. *Proposed by L. R. Galebaugh, Lebanon, Pennsylvania.*

A vertical tree 124 feet high is standing on a hillside whose angle of declivity is unknown. The tree breaks in such a manner that it does not completely separate at the break and the top reaches the ground 52 feet from the base of the stump, measured along the sloping surface of the hillside. The horizontal distance from the base of the stump to the fallen part is 33 feet. How high is the stump?

SOLUTIONS

Late Solutions

133, 134, 138. *Ward Bouwsma, Calvin College, Grand Rapids, Michigan.*

Two Equivalent Volume Integrals

139. [May 1952] *Proposed by H.J. Hamilton, Pomona College.*

Given a closed, convex curve C, not intersected by the x-axis. Let A be the area which C bounds and V the volume of the solid of revolution obtained by revolving A about the x-axis. Now V is given by each of two integral formulas, one obtained by the "circular disc method" of subdividing V and the other by the "cylindrical shell method." (See any elementary calculus text.) Reconcile these integrals without appealing directly to the concept of volume.

I. *Solution by the Proposer.* The one formula may be represented by the line integral $-\pi \oint_c y^2dx$ and the other by $2\pi \oint_c xy\, dy$. But, since the integral of an exact differential taken around a closed path is, in general, zero, we have $\oint_c d(xy^2) = \oint_c (y^2dx + 2xy\, dy) = 0$, and the conclusion follows. The extension to closed curves C of the type for which "Green's Theorem for the Plane" is usually proved is immediate, provided of course that C is not intersected by the x-axis.

II. *Solution by W. B. Carver, Cornell University.* Let E and F be the points of the curve C having the minimum and maximum abscissas, e and f, respectively, and G and H the points having the minimum and maximum ordinates, g and h, respectively. Also let

$y = f_1(x)$	be the equation of the arc	EHF,
$y = f_2(x)$	" " " " "	EGF,
$x = \phi_1(y)$	" " " " "	GFH, and
$x = \phi_2(y)$	" " " " "	GEH.

We are required to show that

$$\int_e^f \pi[f_1^2(x) - f_2^2(x)]dx = \int_g^h 2\pi y[\phi_1(y) - \phi_2(y)]dy$$

With slight restrictions on the function $F(x, y)$ we have the *double* integral over the region

$$\int_R F(x, y)\,dA$$

equal to each of the *iterated* integrals

$$\int_e^f dx \int_{f_2(x)}^{f_1(x)} F(x, y)\,dy \quad \text{and} \quad \int_g^h dy \int_{\phi_2(y)}^{\phi_1(y)} F(x, y)\,dx.$$

(See Franklin's Advanced Calculus, p. 364).

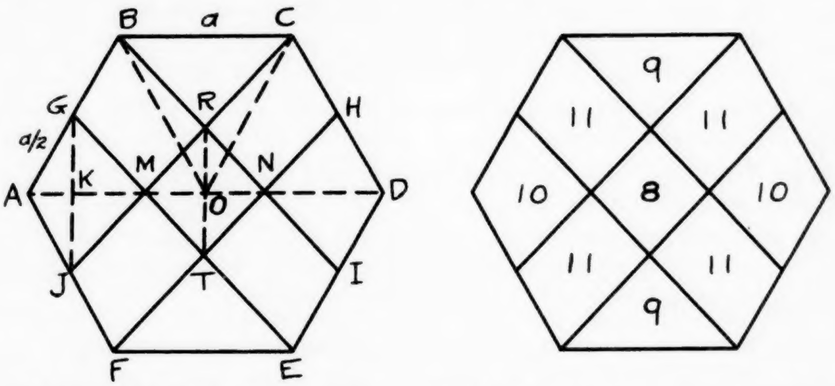
If we take $F(x, y)$ as $2\pi y$ and perform the first integration in each iterated integral we obtain the desired equality between the x and y integrals.

Also solved by *R. F. Reeves, Columbus, Ohio* (2 solutions).

Rhombus Interior to Regular Hexagon

140. [Sept. 1952] *Proposed by R. B. Herrera, Los Angeles City College.*

ABCDEF is a regular hexagon. The midpoints of sides AB, CD, DE, FA are joined respectively to points E, F, B, C forming a small rhombus at the center of the hexagon. Express the area of this rhombus in terms of a side of the hexagon. [This is problem 15 of the May 16, 1952 William B. Orange Mathematics Prize Competition.]



I. *Solution by V. C. Bailey, Evansville College, Evansville, Indiana.* Let the side of the hexagon be a. In the figure, since $AG = GB = IE = a/2$, and GB is parallel to IE, it follows that BI is parallel to GE. Similarly, CJ is parallel to HF. From symmetry, it is obvious that $MRNT$ is a rhombus and that AD passes through MN. Then $AM = MN = ND = AD/3 = 2a/3$ and $MO = a/3$.

Now $GK = a\sqrt{3}/4$ and $AK = a/4$, so $KM = 5a/12$. Then from the similar triangles GJM and RTM we have $RT = (GJ)(MO)/(KM) = 2a\sqrt{3}/5$. Finally, the area of $MRNT$ equals $(MN)(RT)/2$ or $2a^2\sqrt{3}/15$.

II. *Solution by H. F. Heller, Eastern Illinois State College.* $BC = a$, $MN = AD/3 = 2a/3$, hence the areas of the similar triangles MRN and CRB are in the ratio $4/9$. It follows that the altitudes of triangles CRB and COB are in the ratio $3/5$, which is the ratio of their areas, since they have a common base. Hence, (area MRN) $= 4$(area COB)$/15 = 4(a^2\sqrt{3}/4)/15$, and area $MRNT = 2a^2\sqrt{3}/15$.

Also solved by *Leon Bankoff, Los Angeles, California; Eugene Barston, North Hollywood, California; H. H. Berry, U. S. Army Corps of Engineers; Robert Bonic, University of Chicago; Ward Bouwsma, Calvin College, Grand Rapids, Michigan; W. T. Cleagh, Jacksonville, Florida; A. L. Epstein, Geophysical Research Directorate, Boston, Massachusetts; Howard Eves, State University of New York; Peter Gottlieb, Hollywood High School, California* (2 solutions); *Arthur Gregory, Albuquerque, New Mexico; Richard Grote, Van Nuys High School, California; J. E. Hagopian, San Fernando High School, California; Harry Hebb, Port Colborne, Ontario, Canada; Robert Herrmann, Baltimore Polytechnic Institute, Maryland* (2 solutions); *Vern Hoggatt, Oregon State College; Norman Johnson, Carleton College, Northfield, Minnesota; Allen Kirchberg, Charleston, Illinois; Alvin Kopania, Boys' Technical High School, Milwaukee, Wisconsin; Prasert Na Nagara, College of Agriculture, Thailand; W. F. Old, Hofstra College, Hempstead, New York; W. R. Ransom, Reading, Massachusetts; David Rappaport, Chicago, Illinois; L. A. Ringenberg, Eastern Illinois State College; S. H. Sesskin, Hofstra College, Hempstead, New York; J. N. C. Sharp, Upper Canada College, Toronto; A. Sisk, Maryville, Tennessee; W. B. Starbird, Los Angeles Valley Junior College; Charles Stone, North Hollywood High School, California* (2 solutions); *V. C. Throckmorton, Los Angeles City College; M. R. Watson, San Fernando High School, California; S. N. Wilson, Northwest Nazarene College, Nampa, Idaho* (2 solutions); *C. D. Withers, Van Nuys High School, California;* and *the proposer.*

Ringenberg observed that if the area of the hexagon is 90, then the parallel lines divide it into areas of 8, 9, 10 and 11 as shown in the figure. The rows and columns each total 30, so that the lateral boundaries of a "runway" trisect the hexagon.

Palindromic Numbers

141. [Sept. 1952] *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Let $abcd$, $a > 0$, be four-digit integers such that $bc < 99$ is a multiple of 9 and $a + d = 10$. Prove that for $n > 3$ all integers $(10^n - 1)\,abcd$ are palindromes.

Solution by Leon Bankoff, Los Angeles, Calif. If $bc < 99$ is a multiple of 9, then $b + c = 9$. Since $a + d = 10$, we have

$$abcd = 10^3 a + 10^2 b + 10c + d$$

$$= 10^3(10 - d) + 10^2(9 - c) + 10(9 - b) + (10 - a)$$

$$= 10^4 - [10^3(d-1) + 10^2c + 10b + a] = 10^4 - ecba, \text{ where } e = d-1.$$

It follows that

$$(10^n - 1)\ abcd = 10^n abcd - 10^n + 10^n - 10^4 + 10^4 - abcd$$
$$= 10^n abce + (10^{n-4} - 1)10^4 + ecba.$$

For $n > 3$ this is obviously a palindrome in which the central portion consists of $n - 4$ nines.

For $n = 3$, we have $(10^3 - 1)\ abcd = 10^4 abc + 10^3(2d - 11) + cba$ $= 10^4 abc + 10^3(9 - 2a) + cba$, a palindrome for $a \leqq 4$.

For $n = 2$, we have $(10^2 - 1)abcd = 10^4 ab + 10^3(c + d - 10) + 10^2(d + c - 10)$ $+ ba = 10^4 ab + 10^3(9 - a - b) + 10^2(9 - a - b) + ba$, a palindrome for $a + b \leqq 9$.

Also solved by *Ward Bouwsma, Calvin College, Grand Rapids, Michigan; Howard Eves, State University of New York; Arthur Gregory, Albuquerque, New Mexico; M. S. Klamkin, Polytechnic Institute of Brooklyn; Prasert Na Nagara, College of Agriculture, Thailand; and the proposer.*

Gregory noted the necessary additional restriction, $bc > 0$. He also observed that if $N = \sum_{i=0}^{2k-1} 10^i a_{i+1}$ be $2k$-digit integers with $a_{2k} > 0$ and such that $a_1 + a_{2k} = 10$ and $a_i + a_{2k+1-i} = 9$, $(i = 2, 3, \cdots, k)$, then $(10^n - 1)N$ are palindromes for $n > 2k - 1$.

Probability in an Inspection Procedure

142. [Sept. 1952] *Proposed by George Pate, Gordon Military College, Georgia.*

A lot contains n articles. If it is known that r of the articles are defective, and that the articles are inspected in random order, one at a time, what is the probability that the k-th article $(k \geqq r)$ inspected will be the last defective one in the lot?

Solution by W. W. Funkenbusch, Michigan College of Mining and Technology, Sault Ste. Marie Branch. Let us first inspect $k - 1$ articles. The probability that this group contains $r - 1$ defective articles is $C(r, r-1) \times C(n-r, k-r)/C(n, k-1)$. Now the probability that the next article inspected is the remaining defective one is $1/(n-k+1)$. Therefore, the required probability is

$$\frac{r \cdot C(n-r, k-r)}{(n-k+1)\ C(n, k-1)} \quad \text{or} \quad \frac{r(n-r)!\ (k-1)!}{(k-r)!n!}$$

$$\text{or} \quad \frac{r(k-1)(k-2)\cdots(k-r+1)}{n(n-1)(n-2)\cdots(n-r+1)}.$$

This implies that any particular article is inspected only once.

Also solved by *S. W. Hahn, Winthrop College, South Carolina; E. S. Keeping, University of Alberta, Canada; J. E. Kist, Purdue University; M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.; Prasert Na Nagara, College of Agriculture, Thailand; and William Small, Rochester, N. Y.*

Keeping and *Kist* each found this problem, for numerical values of n and r, as an exercise in S. S. Wilks, *Elementary Statistical Analysis*, page 105.

Circles Connected with Inscriptible Trapezium

143. [Sept. 1952] *Proposed by Leon Bankoff, Los Angeles, California.*

O is the center of the circumcircle of trapezium $ADEC$. AD and CE produced meet in B. The circumcenters of triangles ABC and DBE are O' and O'', respectively. Show that $O'O$ equals the circumradius of DBE and that $O''O$ equals the circumradius of ABC.

Solution by Howard Eves, State University of New York. We note that $O'O$ is perpendicular to AC and $O''O$ is perpendicular to DE. Also, since DE and AC are antiparallel with respect to angle B, BO'' is perpendicular to AC and BO' is perpendicular to DE. Therefore $OO'BO''$ is a parallelogram, whence $O'O = BO''$ and $O''O = BO'$, and the desired result is established.

The analogous problem in three-space can be similarly established. Here we have the following: Let a sphere with center O and passing through the vertices A, B, C of a tetrahedron $DABC$ cut the edges DA, DB, DC in A', B', C', respectively. Let O' and O'' be the centers of the circumspheres of tetrahedra $DABC$ and $DA'B'C'$. Then $O'O$ equals the circumradius of $DA'B'C'$ and $O''O$ equals the circumradius of $DABC$. [See, *The American Mathematical Monthly*, **59**, 250, April 1952.]

Also solved by *G. W. Courter, Baton Rouge, Louisiana; Arthur Gregory, Albuquerque, New Mexico; M. S. Klamkin, Polytechnic Institute of Brooklyn, N.Y.; Prasert Na Nagara, College of Agriculture, Thailand; and the proposer.*

A Race-Track Problem

144. [Sept. 1952] *Proposed by J. S. Cromelin, Clearing Industrial District, Chicago.*

The wind is blowing at 56 feet per second along the diagonal of a square track, around which a plane and a car are traveling in opposite directions. The plane is moving with a uniform air speed, the car with a uniform ground speed. The plane first crosses the car at the windward corner. The fourth crossing occurs at one o'clock. The seventh

crossing occurs at the leeward corner, and the eighth at ten minutes after two, again at the windward corner. How large is the track?

Solution by the Proposer. Denote the windward corner by A, and the other three corners, lettered clockwise along the plane's path, by B, C, D. Let x = side of the square track in miles, y = plane's air speed, w = wind's speed, u = plane's speed from A to C, and v = plane's speed from C to A. All speeds are in m.p.h. Also, let the plane make p circuits while the car makes c circuits.

From the data on the seventh and eighth crossings, the car's speed is v. Then since $u > v$, $p > c$. Now the eighth crossing occurs after 7 circuits of either vehicle relative to the other, so $p + c = 7$.

Both vehicles are traveling for the same time interval, so

$$4xc/v = 2xp/v + 2xp/u.$$

That is, $u = pv/(2c - p)$, so $2c > p$.

Both vehicles start and end their travel at the windward corner, so the number of circuits each makes is an integer. Since $2c > p > c$, it follows that $p = 4$, $c = 3$ and $u = 2v$.

We may now easily locate the various crossings, the first at A, the second midway between C and D, the third two-thirds the distance from A to B, the fourth at D, the fifth one-third the distance from B to C, the sixth midway between D and A, the seventh at C, and the eighth at A. Thus the fourth crossing occurs after the car has gone $5x$. Hence the distance traversed by the car between the fourth and eighth crossings is $3(4x) - 5x$ or $7x$. This distance is traversed in 70 minutes, so $v = 7x/(70/60)$ or $6x$ m. p. h.

From the vector diagrams of the plane's flight with and against the wind, we have

$$y^2 = (w/\sqrt{2})^2 + (2v - w/\sqrt{2})^2$$

and

$$y^2 = (w/\sqrt{2})^2 + (v + w/\sqrt{2})^2.$$

Eliminating y, we obtain $v = w\sqrt{2}$, so that $x = v/6 = w\sqrt{2}/6 = 56(60/88)(99/70)/6 = 9$ miles.

(Note: $99/70 = 1.4142857 \cdots \doteq 1.141421 \cdots \doteq \sqrt{2}$, where the error in the approximation is about 5×10^{-5}.)

Euler's ϕ- Function

145. [Sept. 1952] *Proposed by Leo Moser, University of Alberta, Canada.*

It is well-known that $n = 14$ is the smallest even integer for which $\phi(x) = n$ is insolvable. Show for every positive integer, r, that $\phi(x) = 2(7)^r$ is insolvable.

I. *Solution by L. J. Warren, University of Oregon.* Theorem: $\phi(x) = 2p^r$ has a solution if and only if $2p^r + 1$ is a prime, $r \geqq 1$ and p

a prime > 3.

If $2p^r + 1$ is a prime, then $x = 2p^r + 1$ is a solution.

If $\phi(x) = 2p^r$ has solutions, then we note $x = 2q^s$ or q^s $(s \geqq 1)$, since in every other case it is easily shown that 2^2 divides $\phi(x)$ but does not divide $2p^r$. If $s > 1$, then $\phi(x) = \phi(2q^s) = \phi(q^s) = q^{s-1}(q-1) = 2p^r$ which implies $q = p = 3$. Since by hypothesis $p > 3$, we must have $s = 1$. Now $\phi(2q) = \phi(q) = q - 1 = 2p^r$, whence $q = 2p^r + 1$. Since q is a prime, it follows that $2p^r + 1$ must be a prime.

Since 3 always divides $2(7)^r + 1$, then $\phi(x) = 2(7)^r$ has no solutions.

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* In L. E. Dickson's *History of the Theory of Numbers*, Volume I, page 135, there is a result due to Alois Pichler which states "When q is a prime > 3, $\phi(x) = 2q^n$ is impossible if $p = 2q^n + 1$ is not prime; while if p is prime it has the two solutions p and $2p$." Since $2(7)^r + 1 = 2(6+1)^r + 1 = 6A + 3$ has the factor 3 it follows that $\phi(x) = 2(7)^r$ has no solutions. The reference contains other results of a similar nature by Pichler.

III. *Solution by R. R. Phelps, Student, University of California at Los Angeles.* In *The American Mathematical Monthly*, **53**, 327, June-July 1946, V. L. Klee, Jr. proves that if $m = \prod p_i^{a_i}$ where each prime p_i is of the form $3k_i + 1$, then $\phi(x) = 2m$ has no solutions. Since $7 = 2(3) + 1$, it follows that $\phi(x) = 2(7)^r$ is insolvable.

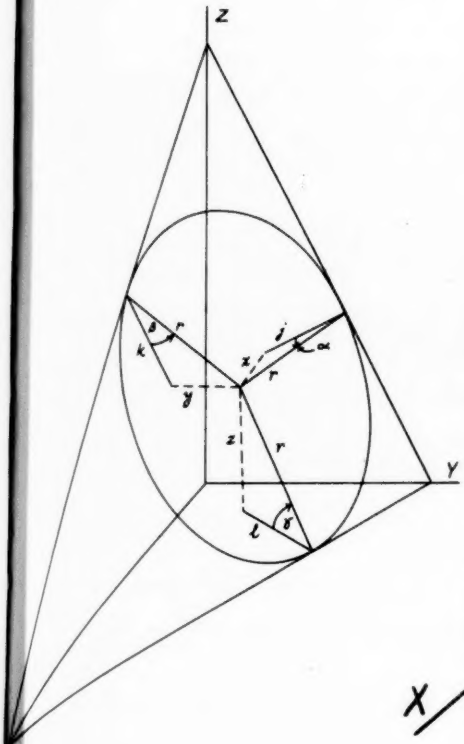
Also solved by *Leon Bankoff, Los Angeles, California; Samuel Holland, Jr., University of Chicago; P. Na Nagara, College of Agriculture, Thailand;* and *the proposer.*

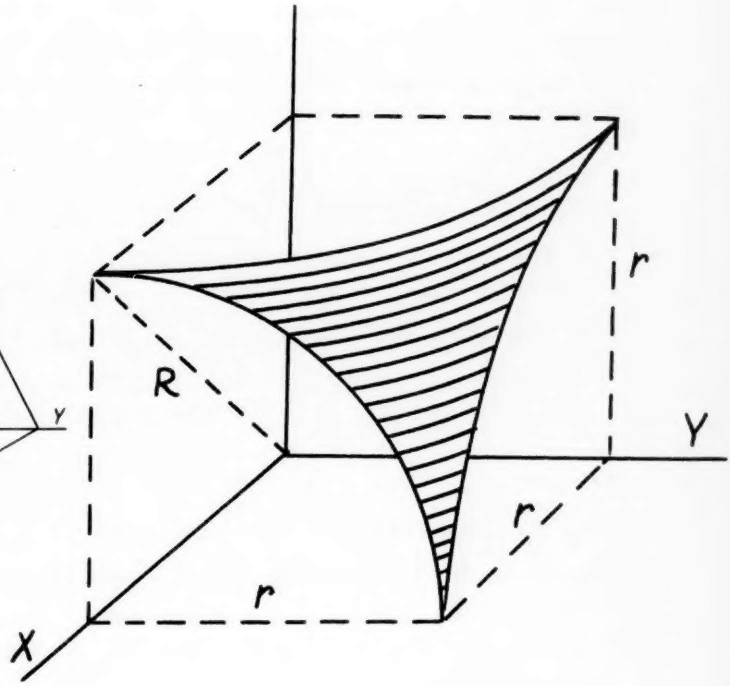
A Spherical Locus

146. [Sept. 1952] *Proposed by A/3C D. L. Silverman, Lowry AFB, Colorado.*

What is the locus of the center of a circle, radius r, which touches each of three mutually perpendicular planes?

Solution by Leon Bankoff, Los Angeles, California. Consider the traces of the plane of the circle on the three mutually perpendicular planes, which are taken as a coordinate system. The radii at the three points of tangency are each perpendicular to one of the traces, as are the projections of these radii on the respective tangent coordinate planes. Consequently, the three radii and their corresponding projections form angles which determine the orientation of the plane in the same manner as the direction angles of the normal do. Indeed, the three dihedral angles between the cutting plane and the coordinate planes are equal to corresponding direction angles of the normal. We therefore may call the dihedral angles made with the YZ, ZX, and XY planes, α, β, and γ, respectively, and the projections upon these planes of the radii to the contact points, j, k, and l, respectively.





Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have

$$j^2/r^2 + k^2/r^2 + l^2/r^2 = 1 \quad \text{or} \quad j^2 + k^2 + l^2 = r^2.$$

Now the center of the circle is $(\sqrt{r^2 - j^2}, \sqrt{r^2 - k^2}, \sqrt{r^2 - l^2})$
so

$$x^2 + y^2 + z^2 = 3r^2 - (j^2 + k^2 + l^2) = 2r^2.$$

Accordingly, the center of the circle lies on the surface of a sphere with radius $R = r\sqrt{2}$. Since the distance of the center of the circle to any of the coordinate planes may equal but not exceed r, the desired locus is the surface of this sphere included between the planes $x = r$, $y = r$, $z = r$, together with congruent surfaces in the other seven octants.

The analogous problem in two dimensions requires the locus of the midpoint of a line of length c which slides so that its extremities move along two perpendicular lines. This locus is a circle with radius $c/2$.

Also solved by *J. L. Botsford, University of Idaho* (2 solutions); *William Leong, University of Calif.*; and *A. Sisk, Maryville, Tennessee.*

Botsford remarked that the problem appeared in a William Lowell Putnam Competition several years ago.

FALSIES

A falsie is a problem from which a correct solution is obtained by illegal operations, or an incorrect result is secured from apparently legal processes. For each of the following falsies, can you offer an explanation? FALSIES will be alternated with TRICKIES and QUICKIES in future issues if our readers show enough interest. Send in your favorite falsies.

F 1. To simplify $[(a+b)^3 + a^3]/[(a+b)^3 + b^3]$ we cancel the exponents to secure $(2a+b)/(a+2b)$. [*Submitted by Norman Anning.*]

F 2. The area of a square with a side of 4 is found by obtaining the sum of the four equal sides, namely 16. [*Submitted by T. C. Wilderman.*]

F 3. Consider the equation $1 - 3 = 4 - 6$. Add 9/4 to each member to secure

$$1 - 3 + 9/4 = 4 - 6 + 9/4$$

or

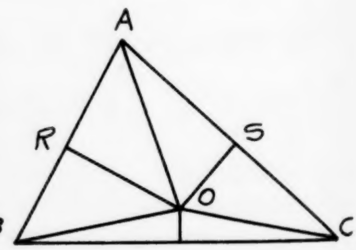
$$(1 - 3/2)^2 = (2 - 3/2)^2.$$

Take the square root of each side to secure $1 - 3/2 = 2 - 3/2$ or $1 = 2$. [*Cecil B. Read in SCHOOL SCIENCE AND MATHEMATICS*, **33**, 587, (June 1933).]

F 4. $\log 1 + \log 2 + \log 3 = \log(1 + 2 + 3) = \log 6$. [*Submitted by Robert Phelps.*]

F 5. $2^3 + 1^3 = 2 \times 3 + 1 \times 3 = 9$. [*David Rosand, Jr., in the Brooklyn Technical High School MATHEMATICS STUDENT*, **21**, 16, (January 1953).]

F 6. To prove that all triangles are isosceles: Let ABC be any triangle. Bisect the angle A by the line AO and let this line meet the perpendicular bisector of BC in O. Draw OS perpendicular to AC and OR perpendicular to AB. Draw OB and OC. Then $RO = OS$, $OB = OC$, $\angle BRO = \angle CSO$, being right angles, and therefore $RB = SC$. Likewise, $AR = AS$. By addition, $AB = AC$. [*From THE PENTAGON*, **7**, 42, (Fall 1947).]



EXPLANATIONS

E 1. This result happens to be true since both the numerator and denominator of the original fraction contain the factor $(a+b)^2 - a(a+b) + a^2$ or $(a+b)^2 - ab$ or $(a+b)^2 - b(a+b) + b^2$.

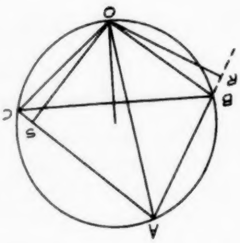
E 2. This is equivalent to $x^2 = 4x$, so that the only two squares for which the perimeter is numerically equal to the area are the null square and the one with a side of 4.

E 3. This familiar fallacy results from ignoring the fact that a quantity has two square roots. It is true that $(1 - 3/2) = -(2 - 3/2)$.

E 4. The result is correct since $1 + 2 + 3 = 1 \times 2 \times 3 = 6$. This constitutes the only solution in positive integers of $x + y + z = xyz$. Other rational solutions are easily obtained from $z = (x + y)/(xy - 1)$, $xy > 1$.

E 5. This is a special case of the identity $a^b + 1^{(a^b - ab + 1)} \equiv a \times b + (a^b - ab + 1)$.

E 6. This fallacious proof is based upon "misdirection" by a faulty figure. The angle bisector either falls along the perpendicular bisector or intersects it exterior to the triangle, on the circumcircle. So that $AB + 2RB = AC$.



QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 81. Factor $x^5 + x + 1$. [*Submitted by M. S. Klamkin.*]

Q 82. Find the sum of the digits appearing in the integers 1, 2, ⋯, $(10^n - 1)$. [*Submitted by Leo Moser.*]

Q 83. In cutting down a tree 6 ft. in diameter, a cut is first made horizontally half way through the tree. A second cut is inclined at an angle of 45° to the horizontal and meets the first cut along a diameter of the tree. Compute the volume of the cylindrical wedge cut out. [*B. E. Meserve and R. E. Pingry in The Mathematics Teacher,* **45**, 261, April 1952.]

Q 84. A circular cylinder of diameter one is tangent internally to a sphere of radius one. Show that the sphere cuts out four square units of area on the cylinder. [*Submitted by J. H. Butchart.*]

Q 85. Two functions of x are differentiable, and not identically zero. Find an example of two such functions having the property that the derivative of their quotient is the quotient of their derivatives. [*Submitted by M. P. Fobes.* This is problem 2, Part II of the March 31, 1951 William Lowell Putnam examination.]

Q 86. Find the center of gravity of a semi-circular wire. [*Submitted by M. S. Klamkin.*]

Q 87. If n points on the circumference of a circle are joined by straight lines in all possible ways, and no three of these lines meet at a single point inside the circle, find the number of triangles formed, all of whose vertices lie inside the circle. [*Submitted by Leo Moser.*]

ANSWERS

A 81. $x^5 + x + 1 = (x^5 + x^4 + x^3) - (x^4 + x^3 + x^2) + (x^2 + x + 1) = (x^2 + x + 1)(x^3 - x + 1)$.

A 82. Pair the integers a and $(10^n - 1 - a)$, $a \geqq 0$. Each pair is easily seen to have a digit sum of $9n$ and the number of pairs is $10^n/2$. Hence the required sum is $9n(10^n/2)$.

A 83. The midsection of the solid is an isosceles right triangle with 3-ft. legs and area of 9/2 sq. ft. Applying the prismoidal formula: $V = 6[0 - 4(9/2) - 0]/6 = 18$ cu. ft.

A 84. The equation of the cylinder is $r = \cos\theta$, and the equation of the sphere is $z^2 + r^2 = 1$. Eliminating r, we have $z = \pm\sin\theta$, $-\pi/2 \leqq \theta \leqq \pi/2$. Hence the area called for can be rolled onto a plane as two arches of the sine curve placed base to base. It is well-known that this area is 4 square units.

A 85. Let the denominator function be x. Then we wish to have $[f(x)/x]' = f'(x)/1 = f'(x)$. Since $f(x)/x$ and $f(x)$ are to have the same derivative, they must differ by a constant. That is, $f(x)/x - f(x) = c$, so $f(x) = cx/(1 - x)$. Therefore, an example of the type desired is $f(x)/g(x)$ where $f(x) = cx/(1 - x)$ and $g(x) = x$.

A 86. Clearly the C.G. lies on the radius perpendicular to the diameter at a distance $\bar{y}$ from the diameter. Rotating the wire about the diameter generates a spherical surface. Hence, by Pappus' theorem we have, $2\pi\bar{y}(\pi r) = 4\pi r^2$. Therefore, $\bar{y} = 2r/\pi$.

A 87. Every set of 6 points on the circumference can be paired in one and only one way such that the three lines joining pairs will form an admissible triangle. Conversely, every admissible triangle has sides leading to 6 points on the circumference. Hence the number of admissible triangles is $\binom{n}{6}$ or $n!/(n - 6)!6!$

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this Department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

CONSTRUCTION OF THE POSITION AND EXTENT OF THE AXES OF AN ELLIPSE, GIVEN A PAIR OF SEMI-CONJUGATE DIAMETERS

In the Nov.-Dec. issue of this magazine S. B. Elrod gives an interesting construction for a pair of special diameters. The following construction for any pair of semi-conjugate diameters may be of interest, and probably not known to many American students. The author obtained this construction from Professor J. W. Bradshaw in his lectures at the University of Michigan.

The basic constructions are shown in Fig. 1.

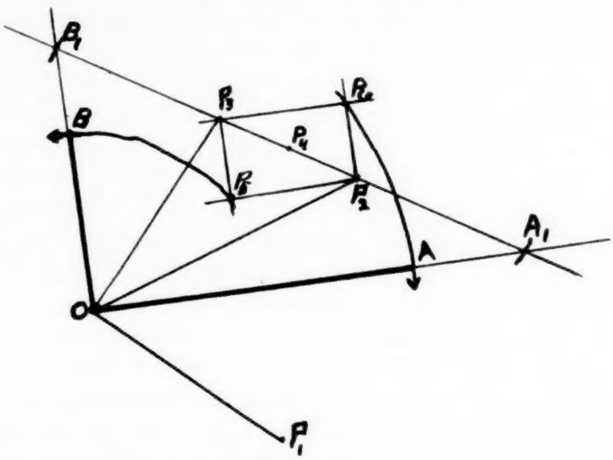


Fig. 1

1. To find the position of the axes.

Let OP_1 and OP_2 be any pair of semi-conjugate diameters. Construct OP_3 equal to and perpendicular to OP_1. Draw the line through P_3 and P_2. Bisect the segment P_3P_2 at P_4. The circle center P_4, radius OP_4, cuts the line through P_2P_3 at A_1 and B_1. Then OA_1 and OB_1 represent the position of the axes.

2. To find the extent of the semi-major and semi-minor axes.

Draw a line through P_2 parallel to OA_1, to cut the line through P_3 and parallel to OB_1 at P_5. Draw the line through P_3 parallel to

OA_1, to cut the line through P_2 and parallel to OB_1 at P_6. The circle, center O, radius OP_6, cuts OA_1 at A; the circle, center O, radius OP_5, cuts OB_1 at B. Then OA and OB are the lengths of the semi-major and semi-minor axes.

The proof of this construction can be determined by analytic methods. For other constructions see "Practical Geometry and Graphic", D. A. Low, Longmans, Green, and Co.; "Projective Geometry", Watson and Watson, Macmillan Co.; "Projective Geometry", Cremona.

C. N. Mills

INFORMATION RETRIEVAL

The sciences present an opportunity and a need for the introduction of an efficient system for the organization of knowledge of a large scale. Many very able people have devoted lifetimes to study in many fields and hundreds of thousands of papers and articles have been published - some of which are immediately recognized as important, some of which doubtless are trivial, or not new, and many of which have real importance which can not be appreciated at the time of publication. The impossibility of one worker being well informed on recent and past developments in "Mathematical Analysis", for instance, or even on a smaller topic such as "Calculus of Finite Differences", is apparent. A partial answer is given by a high degree of specialization, but aside from the disadvantages of possibly undue narrowness, it is nearly impossible to be really well informed on even a very narrow specialty; language barriers, in particular, pose major problems. Leaving aside the relatively unimportant aspect of giving credit for new work, the great difficulties of "information retrieval" in science lead to an enormous amount of duplicated work and wasted effort.

I have recently come in contact with a system for information retrieval called ZATOCODING and I feel that a system using analogous coding principles may be the best hope for alleviating the problem of "lost knowledge". ZATOCODING operates with edge-notched punched cards - one card per document or item to be stored. Each item is characterized by several "descriptors" which should be as broad and general as Each descriptor is then given a punch-code of several notches. In its usual form, ZATOCODING uses single field superimposed random codes, but under certain conditions it is advantageous to use one or more subfields as well. The cards of course contain in print at least the name and location of their respective documents, perhaps together with a brief outline of the subject matter. When it is desired to search the file for information about a given subject, several descriptors (from the file's "descriptor dictionary") appropriate to the subject are selected, and cards are selected from the file on the *logical product* of the search descriptors. If the descriptors are relatively

broad and selection accomplished by using many of them in combination, the file may be searched for relationships not foreseen by the person who did the filing. This is not the case, for instance, in indexes or card catalogs, where the person creating a file must try to anticipate all simple useful interrelationships in his information. Further, information contained in digressions could also be coded even though it is far removed from the main theme of the paper. Machinery for the fully automatic selection of cards at high speeds seems possible using parallel rather than sequential card reading. Perhaps the most intriguing ideas suggest the use of magnetic tapes rather than cards to contain the code items. Files of several million items would seem quite practical.

As a rather narrow possible example, consider a file of information on "functional equations". Each punched card (or its equivalent) would represent either a specific equation or a class of equations for which the solution was available. A few of the descriptors used might be: "Differential", "Integral", "Difference", "linear", "order higher than 3", "Eigenvalues", "ordinary", "n simultaneous equations", etc. Bessel's equation would be coded with "Differential", "2nd order", "ordinary", "linear", "homogeneous" and then perhaps other descriptors describing the coefficients, their singularities, etc. Specifying "Differential", "ordinary", "linear", "homogeneous", "constant coefficient", and "1 dependent variable" would refer always to the one card containing references to the general solution of such equations, no matter what order, for instance, had been specified by the searcher. Much wider applications also seem practical.

In any non-specialized file, a compromise would have to be reached between including very broad fields of knowledge, desirable from the point of view of information retrieval, and including only a narrow subject, which makes for smaller files and faster and easier search. Files should overlap; it may be highly desirable to search in a borderline region. One very appealing aspect of magnetic tapes rather than cards is that with a card the number of punch positions, and so the number of descriptors, is quite limited, whereas with tapes "punch fields" may be as large as desired, seemingly allowing for the possibility of eventually organizing nearly all scientific literature in one file, with great advantages to workers in borderline territories.

The difficulties involved in organizing past and present scientific literature for efficient information retrieval are great, but the advantages are so enormous that I feel certain some solution, perhaps of the sort suggested here, must be attempted in the near future.

Haverford College John Lamperti

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Theory of Probability. By M. E. Munroe, McGraw-Hill, New York, 1951. 213 pp. plus viii, $4.50.

If you are interested in the basic ideas of mathematical probability and have had a year of calculus, this book is for you. Starting with a review of permutations and combinations, Professor Munroe will lead you through distribution functions, stochastic variables and moments to the fundamental limit theorems. These limit theorems, the strong and weak laws of large numbers, the normal and Poisson distributions, are those basic in modern probability and in statistical analysis.

The author's style is friendly, yet distinctly more mature than that of his prerequisite, the usual first year calculus book. The theorems are clearly stated sith more than the usual attention drawn to the places where the assumed mathematical techniques are not sufficient. For example, at one of the more subtle points we are told, "It is only *measureable* subsets we wish to consider. This is no place to consider measureability." This may be confusing at first, but once understood is welcomed. A good use of counter examples is made to illustrate points such as the difference between the strong and weak laws of large numbers. The reader should receive a satisfying knowledge of the basic ideas, with a decent awareness that sometimes the theorems do not apply, and that for full understanding he must continue his study.

There are a large number of very interesting problems, with some step-by-step directions for fitting a mathematical model to a practical situation. On the other hand, there are practically no references to the significance of probability in statistical inference, or its role as an aid to making decisions. While many will regret this omission, the author is able to include many significant topics ordinarily omitted such as the law of the iterated logarith and the error in the normal approximation to the Bernoulian sequence of trials.

The author is to be thanked for bringing these basic ideas of mathematical probability to us in a form suitable for the beginning advanced

student. The book should make an excellent text for classes where applications are not stressed, and a very good supplement for those statisticians who desire an understanding of the mathematical bases of their field.

Paul B. Johnson

The Algebra of Vectors and Matrices. By T. L. Wade. Addison-Wesley; Cambridge, Mass. 1951; pp. ix + 189; $4.50.

It is assumed that the reader of this book has some knowledge of analytic geometry and the elementary properties of determinants as well as a certain amount of mathematical maturity. Apart from these conditions, the book might be described as but little more than elementary. The treatment is detailed with numerous examples and problems to permit the student to consolidate his knowledge as he progresses. It would be suitable as a text for either the prospective mathematician or for one who is interested in learning about the matters concerned with a view to applications.

Vectors are introduced through the geometry of two and three dimensional space and then generalized to n dimensions. The concepts of linear dependence, base, and scalar product of vectors are developed. The matrix algebra includes the characteristic equation, Cayley-Hamilton theorem, rank, and reduction to canonical forms. The author ties this development in with the study of linear transformations, theory of linear equations, bilinear and quadratic forms. The concepts of group, ring, integral domain, and isomorphism are introduced early in the book and used judiciously throughout.

Carman E. Miller

Tensor Analysis, Theory and Applications (Applied Mathematics Series); By I. S. Sokolnikoff; John Wiley and Sons; New York 1951; pp. ix + 335; $6.00.

This book presents an introduction to tensor analysis and to its applications on the junior graduate level.

The first chapter is devoted to a study of linear vector spaces, matrices and linear transformations, and the problem of reduction of quadratic forms. In the second chapter tensor algebra and calculus is developed in the classis manner. A metric is introduced by a positive definite quadratic form $ds^2 = g_{ij}dx^i dx^j$. The resultant space R_n is called Riemannian, and it is shown that the necessary and sufficient condition for it to be Euclidean is that the Riemann-Christoffel tensor be identically zero. The next chapter uses tensor calculus to study the differential geometry of curves and surfaces in Euclidean 3-space. There is an introduction to the calculus of variations, and geodesics in

R_n are thus introduced. The fourth chapter on analytic mechanics contains discussions of Lagrange's equations, Hamilton's principle, the principle of least action, and Hamilton's canonical equations. A brief and intensive survey of relativistic mechanics constitutes chapter five. The final chapter is devoted to the development of the essentials of the mechanics of continuous media. There are some 120 problems.

The best evidence of the great economy of thought gained by the use of tensors and also of the excellence of the author's presentation is the fact that the above material is all clearly and adequately given in 335 pages.

Carman E. Miller

Calculus, Revised Edition. By Joseph Vance McKelvey, vi plus 405 pp. $4.50. The Macmillan Company, New York, 1951.

This book, a revision of an introductory text in calculus, gives to the first-year student a careful presentation of the fundamental concepts of differential and integral calculus. In his development of these concepts the author keeps in mind that the book is intended for the first-year student in calculus. In Chapter II he gives a brief introduction to the topic of limits, in later chapters he makes use of the theorems already introduced, and in the final chapter he discusses the theory of limits in some detail.

The revised edition follows the pattern of the earlier text, but some explanatory material has been amplified, new problems replace many of the old, and new sections on curvilinear motion and the theory of limits have been added.

In differentiation the usual work of an introductory course is presented. Integration first appears in Chapter VI as the inverse operation to differentiation and applications are briefly made to areas, volumes and uniform motion. The techniques of integration are thoroughly treated in Chapters XII and XIII before the summation concept and the definite integral are introduced in Chapter XIV.

Answers to problems on formal differentiation and integration are usually placed in the back of the book while other answers are given in the body of the text.

Because of his belief that "ideas precede vocabulary in the learning process" the author has attempted throughout the text "to describe and to illustrate new ideas first and to give definitions, terminology, and theorems afterwards". By this procedure he has written a readable and teachable text.

Helen G. Russell.